3. Basic notions of the differential geometry of surfaces

Regular C2-surface

Let $\vec{R}$ (u, v) be a parametric representation of a C2-surface F

$$\overrightarrow{R(u,v)} = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} \qquad \text{with its area element } dS = \left\| \vec{R}_u \times \vec{R}_v \right\| \qquad (1)$$

The surface is called a regular surface if $\vec{R}_u \times \vec{R}_v \neq 0$ for all (u,v)

Then for any point $\overrightarrow{R\left(u_P, v_P\right)}$ of the surface F it exists the unit normal vector

$$\overrightarrow{N(u,v)} = \frac{\vec{R}_u \times \vec{R}_v}{\left\| \vec{R}_u \times \vec{R}_v \right\|} \qquad (2)$$

Fundamental Forms

The differential vector $\overrightarrow{dR} = \vec{R}_u \cdot du + \vec{R}_v \cdot dv$ is tangent to the surface in $\overrightarrow{R(u,v)}$

The quantity

$$I = \overrightarrow{dR} \cdot \overrightarrow{dR} = E \cdot du^2 + 2\ F \cdot dudv + G \cdot dv^2 \qquad (3)$$

with $E = \vec{R}_u \cdot \vec{R}_u$, $F = \vec{R}_u \cdot \vec{R}_v$, $G = \vec{R}_v \cdot \vec{R}_v$

is called the first fundamental form of the surface F. It is locally invariant and together with the second fundamental form it determines uniquely the surface F.

The second fundamental follows from a consideration of the unit normal vector (2).

As $d\left(\vec{N} \cdot \vec{N}\right) = 2 \cdot \overrightarrow{dN} \cdot \vec{N} = 0$ the differential vector $\overrightarrow{dN}$ is, as $\overrightarrow{dR}$, tangent to the surface in $\overrightarrow{R(u,v)}$.

For $\overrightarrow{dN}$ we have

$$\overrightarrow{dN} = N_u \cdot du + N_v \cdot dv$$

Multiplication with $-\overrightarrow{dR}$ gives

$$II = -\overrightarrow{dR} \cdot \overrightarrow{dN} = L \cdot du^2 + 2\ M \cdot dudv + N \cdot dv^2$$

wherein (4)

$$L = -\overrightarrow{R_u} \cdot \overrightarrow{N_u},\quad M = -\frac{1}{2} * (\overrightarrow{R_u} \cdot \overrightarrow{N_v} + \overrightarrow{R_v} \cdot \overrightarrow{N_u}),\quad N = -\overrightarrow{R_v} \cdot \overrightarrow{N_v}$$

or alternatively

$$L = \overrightarrow{R_{uu}} \cdot \overrightarrow{N},\quad M = \overrightarrow{R_{uv}} \cdot \overrightarrow{N},\quad N = \overrightarrow{R_{vv}} \cdot \overrightarrow{N}$$

Principal Curvatures and Directions

Let $\overrightarrow{t\left(u_P, v_P\right)}$ be a unit tangent vector to F in the point P. The intersection curve of a plane determined by $\overrightarrow{t}$ and the normal $\overrightarrow{N}$ in P with the surface is called normal section. The curvature of the normal section is the normal curvature of F for the direction $\overrightarrow{t}$. If we consider now all the different tangent directions in P , there are two (mutually perpendicular) directions, called principal, for which the normal curvature attains its maximum and minimum values K1 and K2.

The principal curvature are the solutions of the quadratic equation

$$\left(EG - F^2\right) \cdot K^2 - (EN + GL - 2\ FM) \cdot K + \left(LN - M^2\right) = 0 \qquad (5)$$

The principal directions are given by

$$\frac{dv}{du} = -\frac{(L - E \cdot K)}{(M - F \cdot K)} \qquad (6)$$

A surface point at which the two principal curvatures coincide is an umbilic. In this case for the normal curvature holds

$$K_{umb} = \frac{L}{E} = \frac{M}{F} = \frac{N}{G}$$

Mean and Gauss Curvature

As $K_{1,2}$ are invariants

the mean curvature $$H = \frac{1}{2}\left(K_1 + K_2\right) = \frac{EN + GL - 2\ FM}{2\ \left(EG - F^2\right)} \qquad (7)$$

and

the Gauss curvature $$K_G = K_1 \cdot K_2 = \frac{LN - M^2}{EG - F^2} \qquad (8)$$

are invariants too.

Average Surface Power and Surface Astigmatism

As already mentioned in chapter 2 the average surface power is defined as

$$POW = \frac{(n-1)}{2} \cdot \left(K_1 + K_2\right) = (n-1) \cdot H \qquad (9)$$

and expressing the surface astigmatism by the mean and Gauss curvature we obtain

$$AST = (n-1) \cdot \left(K_1 - K_2\right) = 2\ (n-1) \cdot \sqrt{H^2 - K_G} \qquad (10)$$