4. The Minkwitz Theorem

General

The patent US 1 143 316, application filed in 1911 by Augustin Georges Poullain and Darius Henri Julien Cornet, is probably the first patent which proposes the use of an umbilical main meridian for the construction of a progressive surface. This concept seems relatively obvious. If the power increase of a variable focus lens has the inevitable consequence to accept surface astigmatism, so at least the path for central viewing should be exempt of aberrations. Consequently the first designs on the market were characterised by umbilical principal meridians.

In 1963 Günter Minkwitz, Berlin, investigated the behaviour of the surface astigmatism in the lateral regions of an umbilical line [1]. He made the assumption that the surface had a symmetry plane and that the intersection curve of that symmetry plane with the progressive surface was an umbilical line.

4.1 The proof

4.1.1 Umbilic in the symmetry plane

The proof given here follows the argumentation of the Minkwitz publication.

Let $\overrightarrow{r(s)}$ be the principal meridian in the (x,y)-symmetry plane as a function of its arc length.

$\vec{t}$, $\vec{n}$, $\vec{b} = \vec{t} \times \vec{n}$ designate the tangent unit vector, the principal normal unit vector and the binormal unit vector of this plane curve. $\vec{b}$ is a constant vector.

Then

$$\overrightarrow{R(s,\xi)} = \overrightarrow{r(s)} + \xi \cdot \vec{b} + \eta\left(s,\xi^2\right) \cdot \overrightarrow{n(s)} \qquad \text{with } \eta(s,0) = 0 \qquad (1)$$

describes a surface formed by the principal meridian and its orthogonal sections. The ξ^2 dependance of η expresses the surface symmetry (Fig1).

Let $\kappa(s)$ be the curvature of the principal meridian $\overrightarrow{r(s)}$.

Then the ansatz

$$\eta\left(s,\xi^2\right) = \frac{\kappa(s)}{2} \cdot \xi^2 + o_4(s) \cdot \xi^4 + o_6(s) \cdot \xi^6 + \dots\dots\dots \qquad (2)$$

represents a symmetric orthogonal section with a curvature for $\xi = 0$, which is identical to the curvature of the main meridian in the intersection point (easy to check).

With (2) $\overrightarrow{r(s)}$ becomes an umbilic.

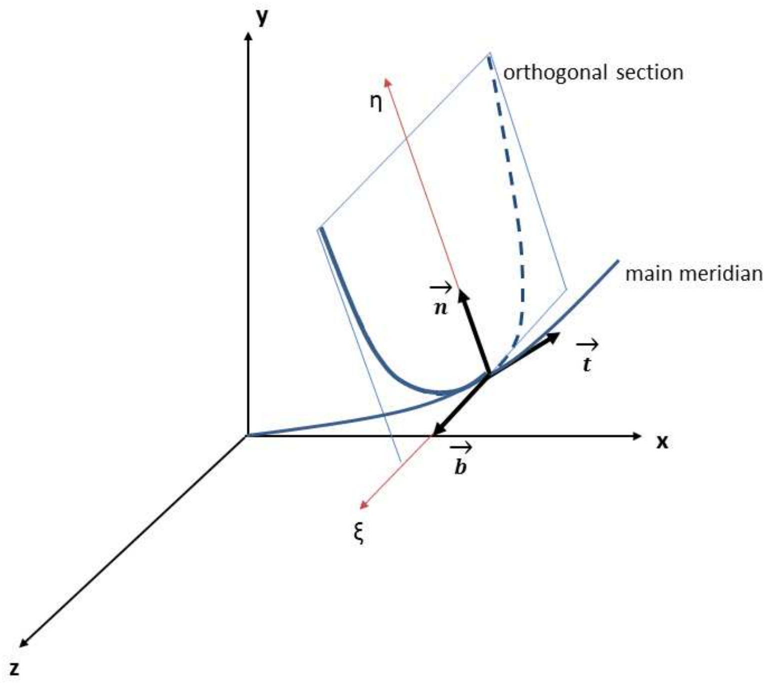

Fig 1

According to chapter 3 equation (10) the surface astigmatism expressed by the mean and Gauss curvature is

$$AST = 2\ (n-1)\ \sqrt{H^2 - K_G} \qquad (3)$$

Now we have to compute H and K_G applying the formulas for the differential geometry of surfaces listed in chapter 3 and using (2) for small ξ.

We have to take into account

$\frac{\mathrm{d}}{\mathrm{d}s}\overrightarrow{r(s)} = \vec{t}$, Frenet's equations: $\frac{\mathrm{d}}{\mathrm{d}s}\vec{t} = \kappa(s)\cdot\vec{n}$, $\frac{\mathrm{d}}{\mathrm{d}s}\vec{n} = -\kappa(s)\cdot\vec{t}$

and after, as G. Minkwitz says, a rather long calculation the results are:

$$H(s,\xi) = -\kappa + \left[\frac{3}{4}\cdot\kappa^{3} - \frac{1}{4}\,\kappa'' - 6\cdot o_{4}\right]\cdot\xi^{2} + \ldots\ldots\ldots$$

(4)

$$K_{G}(s,\xi) = \kappa^{2} + \left[\frac{1}{2}\,\kappa\kappa'' - \kappa'^{2} - \frac{3}{2}\cdot\kappa^{4} + 12\cdot o_{4}\cdot\kappa\right]\cdot\xi^{2} + \ldots\ldots\ldots$$

where the prime sign means the derivative with respect to the arc length s. Taking into account terms until second order it follows

$$H^{2} - K_{G} = \kappa'^{2}\ \xi^{2} + \ldots\ldots\ldots$$

and therefore we obtain for the rate of change of the surface astigmatism perpendicular to the principal meridian

$$\lim_{\xi\to\pm 0}\left(\frac{d}{d\xi}AST(s,\xi)\right) = (+/-)\,2\ (n-1)\cdot\left|\frac{d}{ds}\kappa(s)\right| = (+/-)\,2\cdot\left|\frac{d}{ds}POW(s,0)\right|$$

(5)

This relation is valid for an umbilical principal meridian of a symmetrical progressive surface.

4.1.2 Curved umbilic

With the launch of Gradal HS in 1983 the notion of Horizontal Symmetry was born (regarding the optical characteristics) which meant that the principal meridian had to be curved and therefore it was no line of (geometrical) symmetry anymore. Already in 1976 Alfred Schönhofer, Berlin, analyzed the evolution of the surface astigmatism next to a curved umbilical line [2]. He found that for this more general case

$$grad(AST(s,0)) = (+/-)\,2\ (n-1)\cdot|grad(H(s,0))|\cdot\vec{b}$$

with

$$(n-1)\cdot|grad(H(s,0))| = \sqrt{\left(\frac{d}{ds}POW(s,0)\right)^{2} + \left(\frac{d}{d\xi}POW(s,0)\right)^{2}}$$

wherein ξ is the coordinate perpendicular to the curved meridian.

Generally holds

In the points of an umbilic the surface astigmatism increases linearly and perpendicular to the umbilical line. This increase is two times the gradient of the average surface power in the considered point.

Schönhofer determines for the points on the umbilic also the direction of the principal curvatures, which define the cylinder axis of the surface astigmatism. For the case where the umbilical line is situated in a plane of symmetry, the bisectors between $\vec{t}$ and $\vec{b}$ are the principal directions in the considered point. So the angle between the meridian and the cylinder axis on both sides of the meridian is +/- 45°.

In chapter 5 we check the Minkwitz theorem for the elephant trunk design.

4.2 A non umbilical meridian

In his 1963 publication G. Minkwitz writes that it could be advisable to leave the concept of an umbilic in order to avoid a too strong increase of the peripheral astigmatism.
The following rough calculations for the Varilux 1- type design is not sufficient to validate the foregoing hypothesis but gives some first hints.

Fig 2a shows the lateral increase of the astigmatism for z0=-6, i. e. for the middle of the progression region. When we look at the curve, we observe that the expected linear growth does not only exist in the immediate neighbourhood of the meridian but continues (at least) until 5 mm far from the meridian.

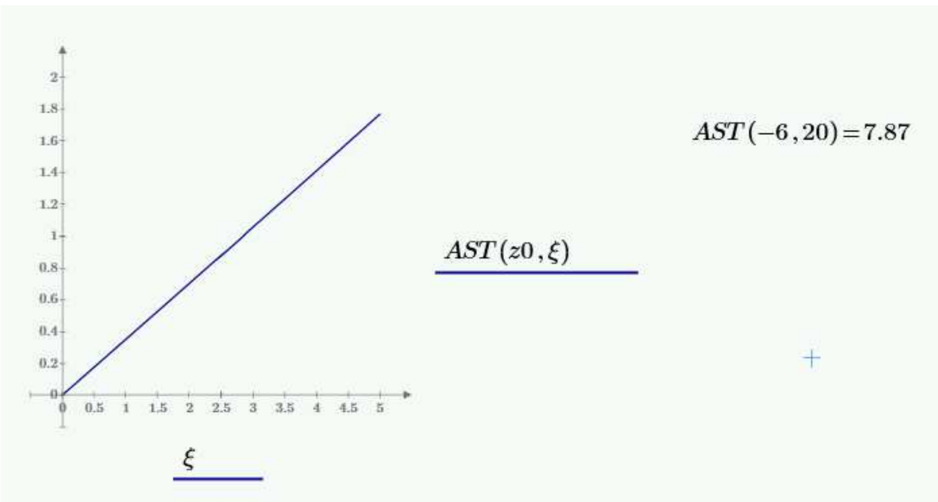

Fig 2a

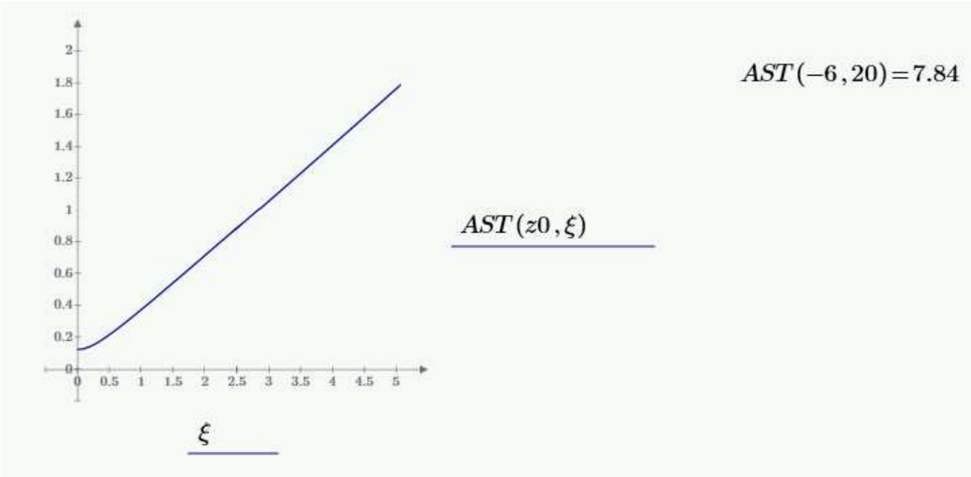

Fig 2b

Fig 2b illustrates the situation where for the Varilux 1- type design for the main meridian a low and physiologically uncritical amount of astigmatism of 0.12 D has been accepted. For $\xi = 0$ we find, as designed, the asigmatism of 0.12 D and a soft transition to the linear increase, so that the curve Fig 2b is differentiable in $\xi = 0$, which is not the case for the graph in Fig 2a.
The lateral astigmatism for $\xi = 20$ is about the same as for the design with an umbilical line.

Making the same analysis for the beginning of the FV part (z0=2) and the NV region (z0=-20) we find similar results: The maximum value for the peripheral astigmatism remains about the same , the increase near to the meridian is a little softer.

Even accepting a physiological critical astigmatism amount of 0.5D on the principal meridian brings only a reduction of 0.1 D of the astigmatism value for $\xi = 20$.

So for the Varilux 1-type geometry, characterised by a principal meridian with stabilized FV- and NV-zones and circles as orthogonal sections, the design accepting astigmatism on the principal meridian has about the same amount of peripheral astigmatism but shows a softer gradient, particularly close to the meridian.
The Varilux 1- type design is a 'hard' design with a rather high level of lateral astigmatism and strong gradients. For a soft geometry, the difference in optical quality between the design with and the design without an umbilical meridian may be greater , particularly with regard to the astigmatism gradient.

References

1. Günter Minkwitz: Über den Flächenastigmatismus bei gewissen symmetrischen Asphären. Optica Acta, vol. 10, no. 3, 1963

2. Alfred Schönhofer: Bemerkungen zu einem Satz von G. Minkwitz über den Astigmatismus asphärischer Flächen, Optica Acta, vol. 23, no. 2, 1976