# 5. The Minkwitz Rüssel

## A first simple design

As already mentioned in chapter 4, the first patent, which seems to propose the use of an umbilic is the patent US 1 143 316 (application 1911) of the French inventors Augustin Georges Poullain, landlord, and Darius Henri Julien Cornet, optician. Their general concept is a progressive design on one ( or even two surfaces) with the second surface serving for the correction of the ametropia. Above and below the progression the main meridian merges into portions of constant curvature, i.e. constant power for the far and near vision zones. What is lacking in the US patent is a description of a machine for grinding surfaces of this type , which according to A.G. Bennett is a part of the original French document in 1910.

We will now try to construct a very simple progressive design with an umbilical line as principal meridian. The main constituent will be the circle. As principal meridian we choose the involute of a circle, also proposed by Poullain and Cornet. The involute is the the path of the end of a string (initially wrapped around a circle ), when it is unwound, always pulling the string taut. As orthogonal sections we choose circles with a radius varying with the radius of curvature of the main meridian at the point of intersection. A.G. Bennett calls this type of design "elephant's trunk", in this publication it will be called the "Rüssel", which means "trunk" in the mother tongue of G. Minkwitz (....and Goethe).

## 5.1 The coordinate system

In Fig 1 the main meridian PM is situated in the yz-plane of a Cartesian coordinate system (x,y,z). Moving from near vision to far vison z increases, the z-axis being tangent to a point in between (for example the center of the finished lens where the prismatic effect is measured).
The orthogonal sections are described in a Cartesian coordinate sytem ($\xi,\eta,\zeta$) moving with the orthogonal section. The point where the section intersects the main meridian is the origin of this coordinate system, the $\zeta$ -axis being tangent to the principal meridian , increasing $\zeta$ meaning increasing z. So in this system the orthogonal section is described by an equation between $\xi$ and $\eta$. $\xi$-and x-values are identical.

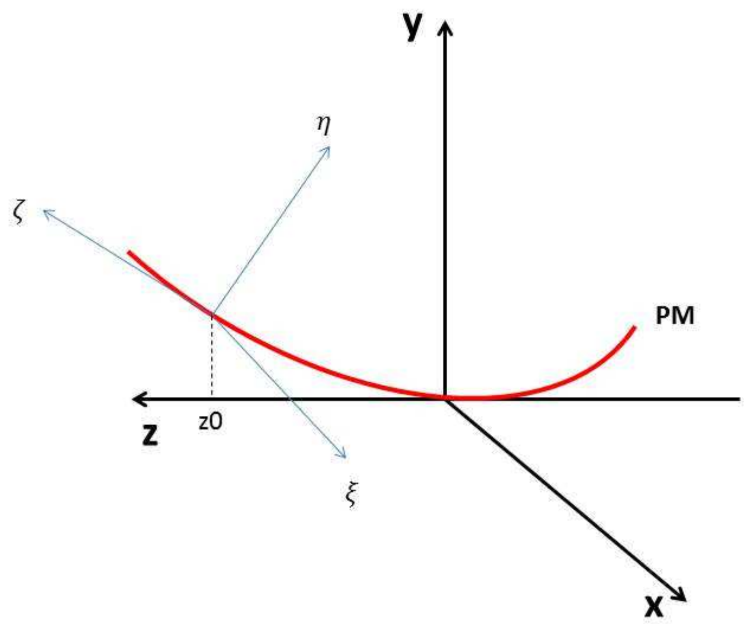


Fig 1

## 5.2 The principal meridian

We take one of the proposals " for the curves in the principal section " in the Poullain/ Cornet-patent " a circular evolvent spiral" i.e. the evolvent or involute of a circle. In a first auxiliary coordinate system (z'',y'') the equation for the involute is

$a := 58.08$ $\qquad$ $t := 0, 0.05 .. 1.60$

$$z''(t) := a \cdot (\cos(t) + t \cdot \sin(t))$$

$$y''(t) := a \cdot (\sin(t) - t \cdot \cos(t))$$

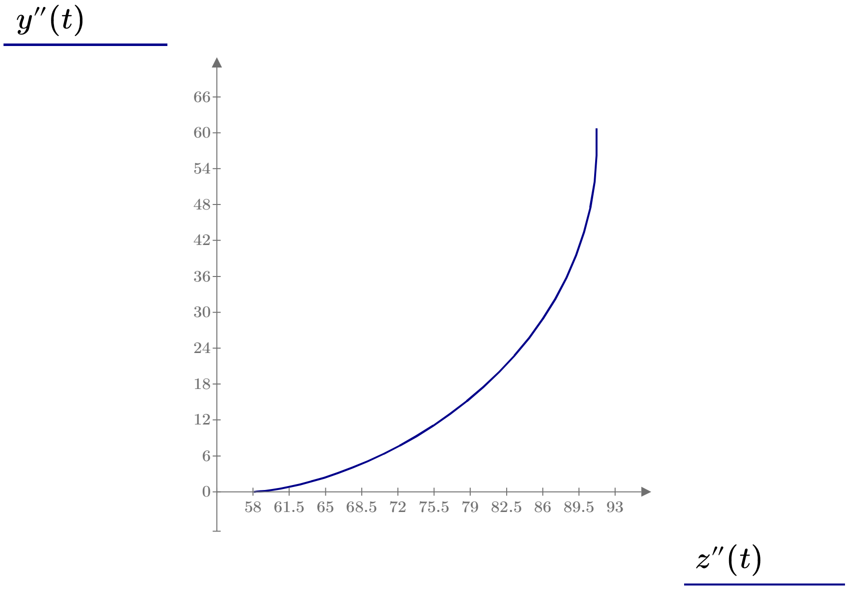


Fig 2

As concerns the power increase along the main meridian we assume that in the far vision reference point ( point where the FV power is measured ) the refractive surface power is 6 D and the power in the NV reference point is 8 D (Add 2 D), corresponding to the curvature radii rf=87.5 mm and r n=65.6 mm ( index of refraction ne=1.525). The position of the two reference points is chosen as in the Varilux 2 patent US 3 687 528 (fig.22) i.e. the far vision point at z0= 12 mm and the near vision point at z0=-16 mm.

**Curvature**

$$K(t) = \frac{1}{a \cdot t}$$

**Calculation of tf,tn und a**

$$\frac{tf}{tn} = \frac{87.5}{65.63} = 1.33$$

arc-length s (approximated by /zf/+/zn/ )

$$s = \frac{a}{2} \cdot t^2 \qquad a = \frac{87.5}{tf} \qquad a = \frac{65.63}{tn}$$

$$\frac{a}{2} \cdot \left(tf^2 - tn^2\right) = 28 \qquad tf := 1.50 \qquad tn := 1.13 \qquad a := 58.08$$

z0=0 corresponds to t0=1.355 (using once again the approximate arc-length-formula).
Now we translate and rotate the (z'',y'') -system so, that the new z-axis is tangent to the involute for the parameter value t0

$$z''(t) := a \cdot (\cos(t) + t \cdot \sin(t)) \qquad y''(t) := a \cdot (\sin(t) - t \cdot \cos(t))$$

$$t0 := 1.355$$

**The new coordinate system has the following equations**

$$z0(t) = z''(t) \cdot cos(\alpha) + y''(t) \cdot sin(\alpha) - z''u \cdot cos(\alpha) - y''u \cdot sin(\alpha)$$

$$y(t) = -z''(t) \cdot sin(\alpha) + y''(t) \cdot cos(\alpha) + z''u \cdot sin(\alpha) - y''u \cdot cos(\alpha)$$

with

**translation**

$$z''u := z''(t0) \qquad y''u := y''(t0)$$

$$z''u = 89.309 \qquad y''u = 39.882$$

**rotation**

$$DE(t) := \frac{\frac{d}{dt} y''(t)}{\frac{d}{dt} z''(t)} \qquad DE(t0) = 4.562$$

$$\alpha := \operatorname{atan}(DE(t0)) \qquad \alpha grad := 1.32 \cdot \frac{180}{\pi} = 75.63$$

$$z0(t) := z''(t) \cdot \cos(\alpha) + y''(t) \cdot \sin(\alpha) - z''u \cdot \cos(\alpha) - y''u \cdot \sin(\alpha)$$

$$Fm(t) := -z''(t) \cdot \sin(\alpha) + y''(t) \cdot \cos(\alpha) + z''u \cdot \sin(\alpha) - y''u \cdot \cos(\alpha)$$

In this new (z0,y)-coordinate system the main meridian shows the position we were looking for

$$t := 0.85, 0.86 .. 1.70$$

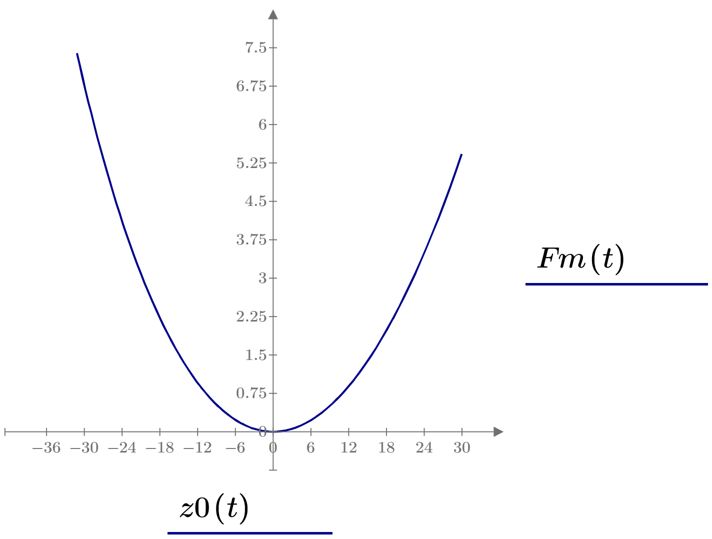


Fig 3

definition of the derivatives

$$D1Fm(t) := \frac{\frac{d}{dt} Fm(t)}{\frac{d}{dt} z0(t)} \qquad D2Fm(t) := \frac{\frac{d}{dt} D1Fm(t)}{\frac{d}{dt} z0(t)}$$

**crosschecking z0 and power P(t) for t0, tf and tn**

$K(t) := \frac{1}{a \cdot t}$ $\qquad P(t) := 525 \cdot K(t)$

$t0 := 1.355 \qquad z0(t0) = 0$

$tf := 1.50 \qquad z0(tf) = 11.98 \qquad P(tf) = 6.03$

$tn := 1.13 \qquad z0(tn) = -16.11 \qquad P(tn) = 8$

**a steady power increase from the top to the bottom**

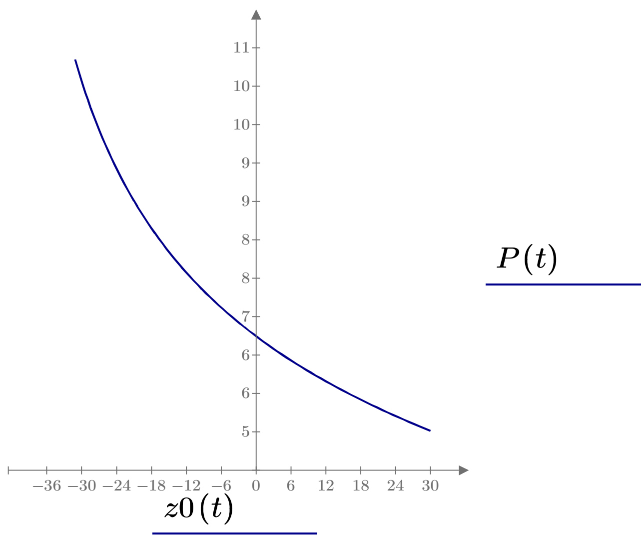


Fig 4

## 5.3 Geometry and equations of the progressive surface

Assuming that the sections are circles with radii of curvature which are respectively equal to the radii of curvature of the meridian at the points of intersection (umbilical line), the vertex equation of the circle is

$$\eta(t,\xi) := \frac{\left(1-\sqrt{1-\xi^2 \cdot K(t)^2}\right)}{K(t)}$$

So in the (x,y,z)-coordinate system we get

$$x(t,\xi) := \xi$$

$$y(t,\xi) := Fm(t) + \frac{\eta(t,\xi)}{\sqrt{1+D1Fm(t)^2}}$$

$$z(t,\xi) := z0(t) - \eta(t,\xi) \cdot \frac{D1Fm(t)}{\sqrt{1+D1Fm(t)^2}}$$

### 5.3.1 Fundamental Forms

Definitions

$$r(t,\xi) := \begin{bmatrix} x(t,\xi) \\ y(t,\xi) \\ z(t,\xi) \end{bmatrix}$$

$$xz0(t,\xi) := \frac{\frac{d}{dt}x(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$yz0(t,\xi) := \frac{\frac{d}{dt}y(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$zz0(t,\xi) := \frac{\frac{d}{dt}z(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$rz0(t,\xi) := \begin{bmatrix} xz0(t,\xi) \\ yz0(t,\xi) \\ zz0(t,\xi) \end{bmatrix}$$

$$x\xi(t,\xi) := \frac{d}{d\xi}x(t,\xi)$$

$$y\xi(t,\xi) := \frac{d}{d\xi}y(t,\xi)$$

$$z\xi(t,\xi) := \frac{d}{d\xi}z(t,\xi)$$

$$r\xi(t,\xi) := \begin{bmatrix} x\xi(t,\xi) \\ y\xi(t,\xi) \\ z\xi(t,\xi) \end{bmatrix}$$

$$xz0z0(t,\xi):=\frac{\frac{d}{dt}xz0(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$yz0z0(t,\xi):=\frac{\frac{d}{dt}yz0(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$zz0z0(t,\xi):=\frac{\frac{d}{dt}zz0(t,\xi)}{\frac{d}{dt}z0(t)}$$

$$rz0z0(t,\xi):=\begin{bmatrix} xz0z0(t,\xi) \\ yz0z0(t,\xi) \\ zz0z0(t,\xi) \end{bmatrix}$$

$$x\xi\xi(t,\xi):=\frac{d^2}{d\xi^2}x(t,\xi)$$

$$y\xi\xi(t,\xi):=\frac{d^2}{d\xi^2}y(t,\xi)$$

$$z\xi\xi(t,\xi):=\frac{d^2}{d\xi^2}z(t,\xi)$$

$$r\xi\xi(t,\xi):=\begin{bmatrix} x\xi\xi(t,\xi) \\ y\xi\xi(t,\xi) \\ z\xi\xi(t,\xi) \end{bmatrix}$$

$$xz0\xi(t,\xi) := \frac{d}{d\xi} xz0(t,\xi)$$

$$yz0\xi(t,\xi) := \frac{d}{d\xi} yz0(t,\xi)$$

$$zz0\xi(t,\xi) := \frac{d}{d\xi} zz0(t,\xi)$$

$$rz0\xi(t,\xi) := \begin{bmatrix} xz0\xi(t,\xi) \\ yz0\xi(t,\xi) \\ zz0\xi(t,\xi) \end{bmatrix}$$

1.Fundamental Form and unit normal vector

$$E(t,\xi) := rz0(t,\xi) \cdot rz0(t,\xi)$$

$$F(t,\xi) := rz0(t,\xi) \cdot r\xi(t,\xi)$$

$$G(t,\xi) := r\xi(t,\xi) \cdot r\xi(t,\xi)$$

$$No(t,\xi) := \frac{rz0(t,\xi) \times r\xi(t,\xi)}{\|rz0(t,\xi) \times r\xi(t,\xi)\|}$$

2. Fundamental Form

$$L(t,\xi) := rz0z0(t,\xi) \cdot No(t,\xi)$$

$$M(t,\xi) := rz0\xi(t,\xi) \cdot No(t,\xi)$$

$$N(t,\xi) := r\xi\xi(t,\xi) \cdot No(t,\xi)$$

$$H1(t,\xi) := E(t,\xi) \cdot N(t,\xi) + G(t,\xi) \cdot L(t,\xi) - 2 \cdot F(t,\xi) \cdot M(t,\xi)$$

$$H2(t,\xi) := E(t,\xi) \cdot G(t,\xi) - F(t,\xi)^2$$

$$H3(t,\xi) := L(t,\xi) \cdot N(t,\xi) - M(t,\xi)^2$$

Principal curvatures

*the root expression is put between an absolute value sign . The root expression is close to zero near the umbilic (see values below) and so the iterations of the numerical evaluation can result in slightly negative values.*

$$K1(t,\xi) := \frac{\left(H1(t,\xi) + \sqrt{\left\|\left(H1(t,\xi)^2 - 4\ H2(t,\xi) \cdot H3(t,\xi)\right)\right\|}\right)}{2 \cdot H2(t,\xi)}$$

$$K2(t,\xi) := \frac{\left(H1(t,\xi) - \sqrt{\left\|\left(H1(t,\xi)^2 - 4\ H2(t,\xi) \cdot H3(t,\xi)\right)\right\|}\right)}{2 \cdot H2(t,\xi)}$$

### 5.3.2 Mean optical power POW and Astigmatism AST

$$POW(t,\xi) := 525 \cdot \frac{(K1(t,\xi) + K2(t,\xi))}{2}$$

$$AST(t,\xi) := 525 \cdot (K1(t,\xi) - K2(t,\xi))$$

Check that the principal meridian is an umbilical line

*as mentioned above the values are practically zero*

$\xi := 0$ $\quad t := 0.8, 0.85 .. 1.6$

$$AST(t,\xi) = \begin{bmatrix} 2.5 \cdot 10^{-7} \\ 2.6 \cdot 10^{-7} \\ 3.9 \cdot 10^{-7} \\ 0 \\ 0 \\ 0 \\ 2.3 \cdot 10^{-7} \\ 1.7 \cdot 10^{-7} \\ 0 \\ 1.7 \cdot 10^{-7} \\ 0 \\ 1.7 \cdot 10^{-7} \\ 1.7 \cdot 10^{-7} \\ 1.7 \cdot 10^{-7} \\ 1.7 \cdot 10^{-7} \\ 1.7 \cdot 10^{-7} \\ 1.6 \cdot 10^{-7} \end{bmatrix}$$

**Lines of constant mean optical power (Isopowerlines )**

Loop calculation

$\xi min := -24$ $\quad \xi max := 24$ $\quad n := 13$ $\quad i := 0 .. n$

$$\Delta\xi := \frac{\xi max - \xi min}{n} \qquad \xi_i := \xi min + i \cdot \Delta\xi$$

$tmin := 1$ $\quad tmax := 1.6$

$$\Delta t := \frac{tmax - tmin}{n} \qquad t_i := tmin + i \cdot \Delta t$$

$$MPOW(\xi, t) := \begin{array}{l} \text{for } i \in 0..\text{last}(\xi) \\ \quad \text{for } j \in 0..\text{last}(t) \\ \quad\quad MPow_{i,j} \leftarrow POW\left(t_j, \xi_i\right) \\ \quad\quad Mz0_{i,j} \leftarrow z0\left(t_j\right) \\ \quad\quad M\xi_{i,j} \leftarrow \xi_i \\ \text{return } \begin{bmatrix} M\xi \\ Mz0 \\ MPow \end{bmatrix} \end{array}$$

$$ISOPOW := MPOW(\xi, t) = \begin{bmatrix} [14 \times 14] \\ [14 \times 14] \\ [14 \times 14] \end{bmatrix}$$

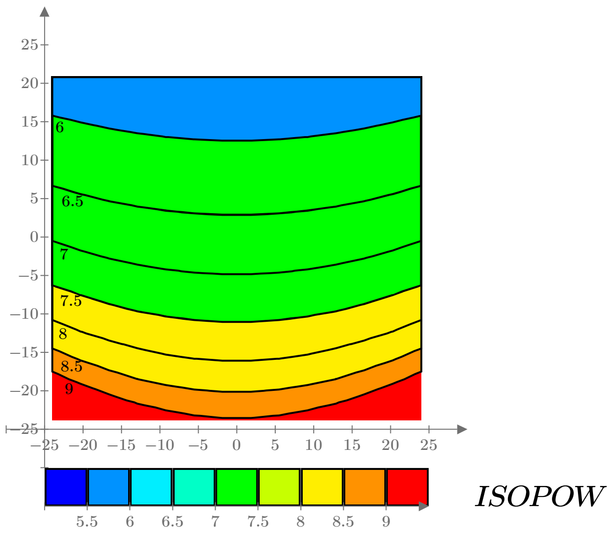


Fig 5

## Lines of constant astigmatism (Isoastigmatism plot)

$\xi min := -24$ $\quad \xi max := 24$ $\quad n := 13$ $\quad i := 0..n$

$$\Delta\xi := \frac{\xi max - \xi min}{n} \qquad \xi_i := \xi min + i \cdot \Delta\xi$$

$tmin := 1$ $\quad tmax := 1.6$

$$\Delta t := \frac{tmax - tmin}{n} \qquad t_i := tmin + i \cdot \Delta t$$

```
MAST(ξ, t) := || for i ∈ 0..last(ξ)
              ||    || for j ∈ 0..last(t)
              ||    ||    || MAsti[i,j] ← AST(t[j], ξ[i])
              ||    ||    || Mz0[i,j] ← z0(t[j])
              ||    ||    || Mξ[i,j] ← ξ[i]
              || return [ Mξ ; Mz0 ; MAsti ]
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$$ISOAST := MAST(\xi, t) = \begin{bmatrix} [14 \times 14] \\ [14 \times 14] \\ [14 \times 14] \end{bmatrix}$$

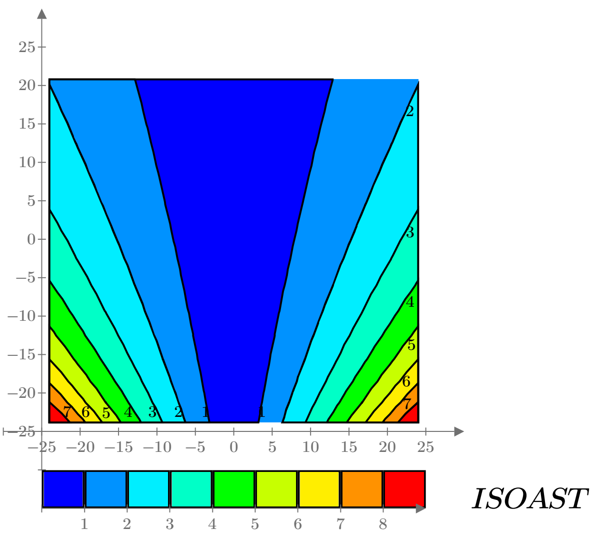


Fig 6

## 5.4 Analysis of design and calculations

The power contour plot Fig 5 shows a power increase with almost horizontally isopower lines , which is exactly what we are looking for. But the progression length of 28 mm is too long for natural physiological conditions of head posture and eye movements. Additionally the steady power growth from top to the bottom of the lens causes a high amount of lateral astigmatism, particularly in the lower part of the lens (Fig 6), where the near vision zone becomes rather narrow. The characteristics of such a design are described by Bernard Maitenaz when he tells the history of the Varilux 1 invention [1] and in his famous "Varilux2- patent" US 3 687 528 (see there Fig 2 and 3). So a major challenge of the Varilux development was the to put in place the manufacturing means for a meridian with stabilized power zones ( see chapter 6).
As the power gradient becomes steeper approaching the near vision zone, the isoastigmatism lines in Fig 6 show a conical shape with decreasing z0-values.This a beautiful qualitative demonstration of the Minkwitz-theorem, but we will try  to check the relation also quantitatively.

$$Minkw(t,\xi) := \frac{\frac{\mathrm{d}}{\mathrm{d}\xi}(AST(t,\xi))}{\frac{\frac{\mathrm{d}}{\mathrm{d}t}POW(t,\xi)}{\frac{\mathrm{d}}{\mathrm{d}t}z0(t)}}$$

$t := 0.9, 1.0 .. 1.7$ $\qquad$ $\xi := 1$

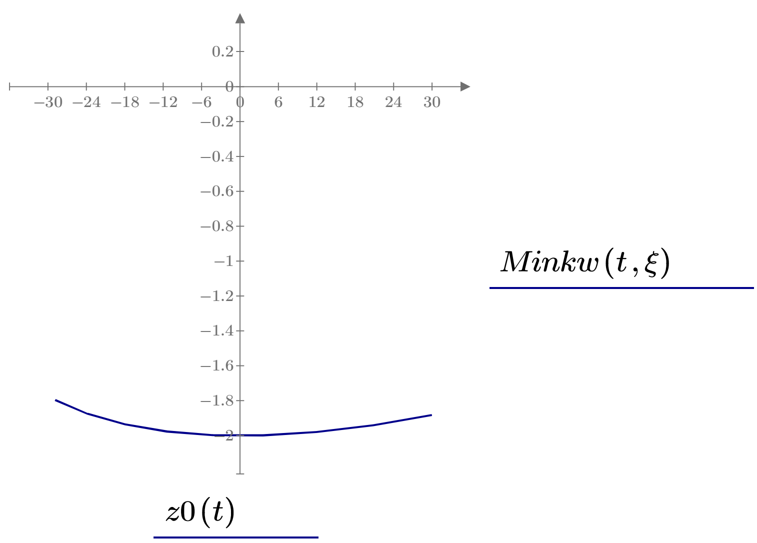


Fig 7

In the z0-region between -25 and 25 mm the ratio Minkw(t, $\xi$) is very close to the theoretical value of 2. The reason for the deviation is, that the exact verification requires $\xi$ =0, but for $\xi$=0 the computing program does not converge , so a $\xi$-value of 1.0 had to be chosen.

The Rüssel represents a variable focus surface with a continuous power increase from top to bottom with no power stabilization for far and near vision.The consequences are an unnecessarily high peripheral astigmatism and rather narrow viewing zones for seeing into the far and near distances.
So a next logical step, in trying to improve the visual performance, would be to modify the backbone of the design, the principal meridian, in a way to ensure large viewing zones for far and near vision.