9. Progressiv R, the" refraction-correct" progressive lens

The background

In the 70's/80's of the last century Rodenstock was ex equo with Essilor one of the leading spectacle manufacturers of the world . After the Varilux invention by Bernard Maitenaz Rodenstock marketed the BBGR progressive lens Zoom . This design however was not competitive compared to Varilux 2 and moreover the renown of Rodenstock called for an own brand.
So G. Guilino and R. Barth started the development of progressive surfaces in Germany and their patent US 4 315 673 is one of the most beautiful examples of a progressive design constituted by analytical functions.
This surface represents a design which is distinctly softer than Varilux 1, but nevertheless distinguishes itself by large viewing zones for far and near vision. As the "refraction correct progressive" it was attacking the (inofficial) recommendation on the market, that for Varilux 2 the optimum comfort is achieved when the far vision power is corrected by -0.25 D and parallel to this the add power is increased by 0.25 D.

9.1 The coordinate system

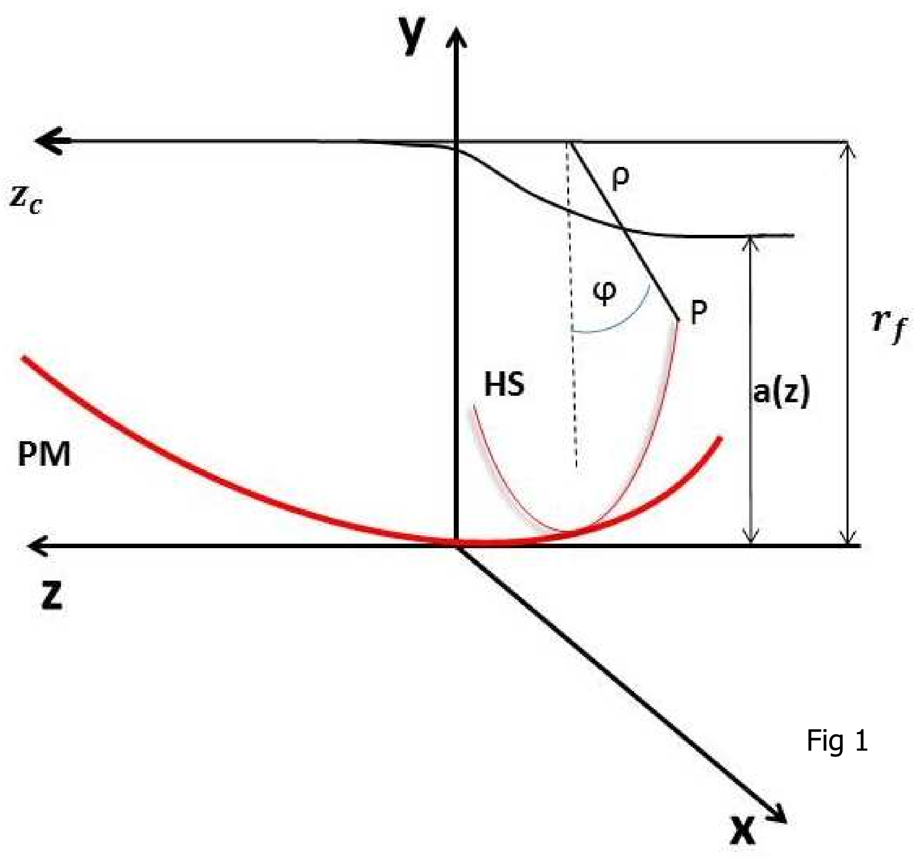


Fig 1

The principal meridian PM is situated in the (yz)-plane of a Cartesian coordinate system (x,y,z) and is described by the function Fm(z). Moving from near vision to far vison z increases, the z-axis being tangent to PM in a point in between (for example the center of the finished lens where the prismatic effect is measured).

For the analytical description of the horizontal section HS (please note, that we specify the horizontal and not the orthogonal section) of the surface the patent uses a cylinder coordinate system (ρ, φ), the cylinder axis z_c being parallel to the z-axis in a distance r_f (radius of curvature in the far reference point).
As we will see in chapter 9.5 the distance "z axis to cylinder-axis" zz_c with value r_f gives a design with a rather large far vision area (with smaller NV) and a distance r_n (radius of curvature in the near reference point) gives a large near vision part (with smaller FV). So in the patent the surface is defined in an auxiliary cylinder coordinate system with a curved cylinder axis a(z), giving a distance zz_c close to r_f for positive z values and close to r_n for negative z.

$$a(z) = \left(\frac{1}{rf} + \frac{\left(1 + e^{3 \cdot (z-1)}\right)^{-30}}{52.5} \right)^{-1}$$

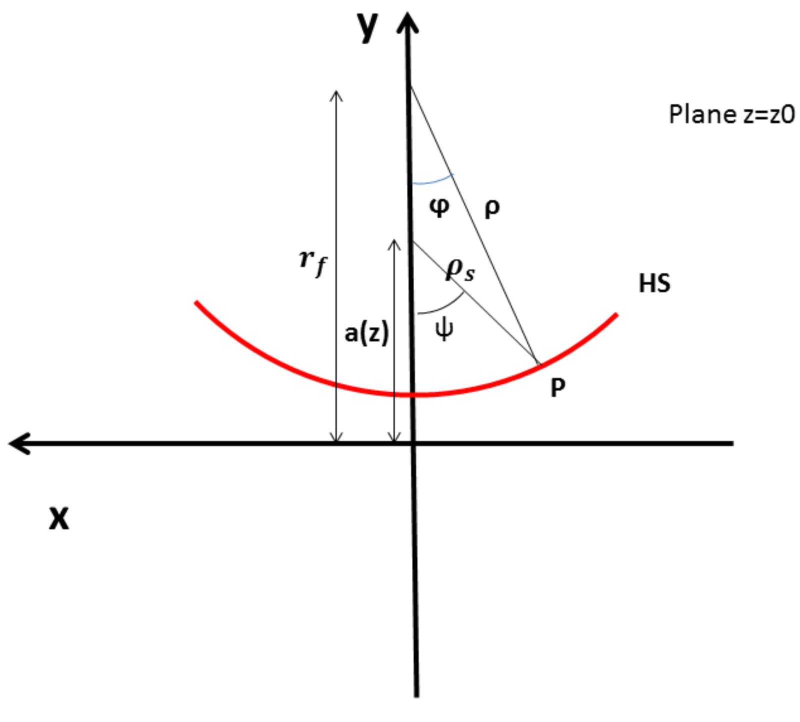


Fig 2

In chapter 9.4 below we will see that the exponential factor c0 (here 3) has a strong impact on the design softness.
The relation between the original coordinate system (ρ,φ) and the auxiliary system (ρs, ψ), looking in the direction of the z axis, is illustrated in Fig 2.

9.2 Description of the progressive surface

ρs is given by a Fourier series

$$\rho s(\psi,z)=\sum_{n=0}^{\propto} bn(z)\ cos(n\cdot k(z)\cdot\psi)$$

where the negative b_n are zero because the surface is symmetrical to the z-axis (to avoid confusions we choose for the coefficients the notation b_n instead of a_n as in the patent).
Taking into account only the terms for n=0 and n=1 and requiring that the main meridian $\rho s(0,z)=f_s(z)$ is an umbilical line we get the equation

$$\rho s(\psi,z)=fs(z)+\frac{1}{k(z)^2}\cdot\left(fs(z)-\frac{fs(z)^2\ Fm''(z)}{\left(1+Fm'(z)^2\right)}\right)\cdot(1-cos(k(z)\cdot\psi))$$

where we will call k(z) the periodicity of the first order Fourier term.

The relation between ρs, ρ, ψ and φ according to Fig 1 and 2 is given by

$$x=\rho(\varphi,z)\cdot sin(\varphi)=\rho s(\psi,z)\cdot sin(\psi)$$

$$y=rf-\rho(\varphi,z)\cdot cos(\varphi)=a(z)-\rho s(\psi,z)\cdot cos(\psi)$$

In their publications and marketing brochures Rodenstock calls the new design the surface with variable periodicity. The horizontal sections of Progressiv R are defined as a periodic functions of the cylinder coordinate φ. The periodicity k(z) varies continuously with the position of the horizontal section.
In the patent, column 5, as an example, the periodicity k(z) is given by the following expression:

$$k(z)=3+\frac{7}{\left(1+e^{-3\cdot(z+1.8)}\right)}$$

According to the patent the rather low k value 3 for high negative z should provide a large near vision zone, the rather high k value 10 for high positive z should guarantee an aberration-free far vision area. This will be discussed below.

We will consider two versions of Progressif R, first, one example given in the patent US 4 315 673 and second, the product which was sold on the market.

9.3 The patent

9.3.1 The principal meridian

From the patent we pick out the design in table 3, columns 7 and 8. The curvature of the main meridian for add 3 is given by claim 20

$$K(z) = \frac{1}{8.75} + \frac{0.03}{0.525}\left(1-\left(1+e^{-3.09\cdot(z+1.82)}\right)^{-30}\right)$$

This is the equation for the geometrical curvature , not the optical power. The meridian shows stabilized curvature in the far vision as well as in the near vision part. The far vision radius is 87.5 mm corresponding to the far vision power of 6 D. In the patent the coefficients in the exponential term depend on the add power. So the transition from the stabilized zones into the intermediate section with varying power can be designed more or less softly and has so an essential influence on the peripheral astigmatism.

The main meridian Fm(z) and its first derivative D1Fm(z) are obtained by integration of the differential equation for the curvature.
(In the Rodenstock patent the dimension of length is given in cm , so we do the same in the following calculations).

Integrating the differential equation for the curvature K(z) of the meridian

$$\frac{Fm''(z)}{\left(1+Fm'^2\right)^{\frac{3}{2}}} = K(z) \qquad Fm(z) = x1(z)$$

$$x1' = x2$$

$$x2' = \left(1+x2^2\right)^{\frac{3}{2}} \cdot K(z)$$

$$K(z) := \frac{1}{8.75} + \frac{0.03}{0.525}\left(1 - \left(1 + e^{-3.09\cdot(z+1.82)}\right)^{-30}\right)$$

$$D(z,X) := \begin{bmatrix} X_1 \\ \left(1 + {X_1}^2\right)^{\frac{3}{2}} \cdot K(z) \end{bmatrix}$$

$$X(z) = \begin{bmatrix} x1(z) \\ x2(z) \end{bmatrix}$$

$$u := 0.3723 \qquad v := 0.3017$$

$$init := \begin{bmatrix} u \\ v \end{bmatrix}$$

u and v are the initial conditions for Fm(z) and D1Fm(z)=Fm'(z) in z=2.5 cm, calculated first from a circle with rf=8.75 cm and then iteratively corrected, so that for z=0 Fm(z) and D1Fm(z) are zero

$$Zi := 2.5$$
$$Zf := -2.5$$
$$N := 100$$

$$sol := \mathrm{AdamsBDF}(init, Zi, Zf, N, D)$$

$$sol = \begin{array}{ccc} \text{z} & \text{x1=Fm(z)} & \text{x2=Fm'(z)} \\ \end{array}$$
$$\begin{bmatrix} 2.5 & 3.723\cdot10^{-1} & 3.017\cdot10^{-1} \\ 2.45 & 3.574\cdot10^{-1} & 2.952\cdot10^{-1} \\ 2.4 & 3.428\cdot10^{-1} & 2.887\cdot10^{-1} \\ 2.35 & 3.285\cdot10^{-1} & 2.823\cdot10^{-1} \\ 2.3 & 3.145\cdot10^{-1} & 2.759\cdot10^{-1} \\ 2.25 & 3.009\cdot10^{-1} & 2.696\cdot10^{-1} \\ & \vdots & \end{bmatrix}$$

Calculating the main meridian as a function of z : Fm(z)

Arranging the data in an ascending order of z

$$data := \mathrm{csort}(sol, 0) = \begin{bmatrix} -2.5 & 4.83 \cdot 10^{-1} & -4.3 \cdot 10^{-1} \\ -2.45 & 4.62 \cdot 10^{-1} & -4.19 \cdot 10^{-1} \\ -2.4 & 4.41 \cdot 10^{-1} & -4.08 \cdot 10^{-1} \\ -2.35 & 4.21 \cdot 10^{-1} & -3.97 \cdot 10^{-1} \\ -2.3 & 4.01 \cdot 10^{-1} & -3.86 \cdot 10^{-1} \\ -2.25 & 3.82 \cdot 10^{-1} & -3.76 \cdot 10^{-1} \\ & & \vdots \end{bmatrix}$$

$$Z := data^{\langle 0 \rangle} \qquad X1 := data^{\langle 1 \rangle} \qquad X2 := data^{\langle 2 \rangle}$$

$$S1 := \mathrm{cspline}(Z, X1)$$

$$Fm(z) := \mathrm{interp}(S1, Z, X1, z)$$

$$S2 := \mathrm{cspline}(Z, X2)$$

$$D1Fm(z) := \mathrm{interp}(S2, Z, X2, z)$$

$$z := -2.5, -2.4 .. 2.5$$

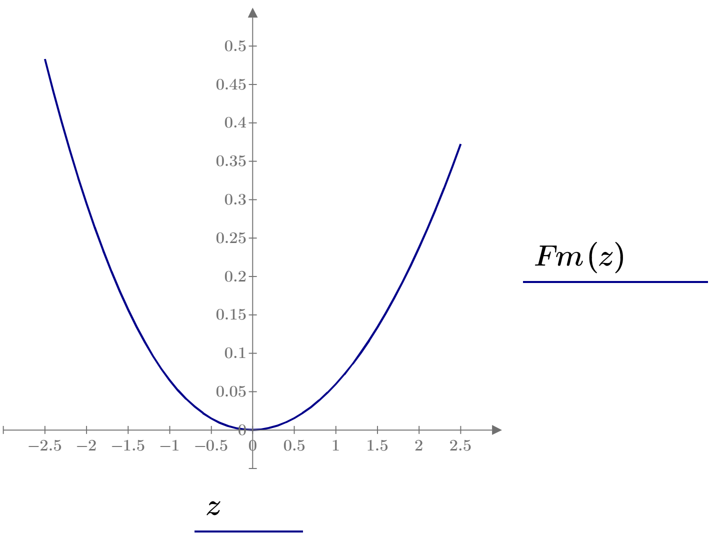


Fig 3

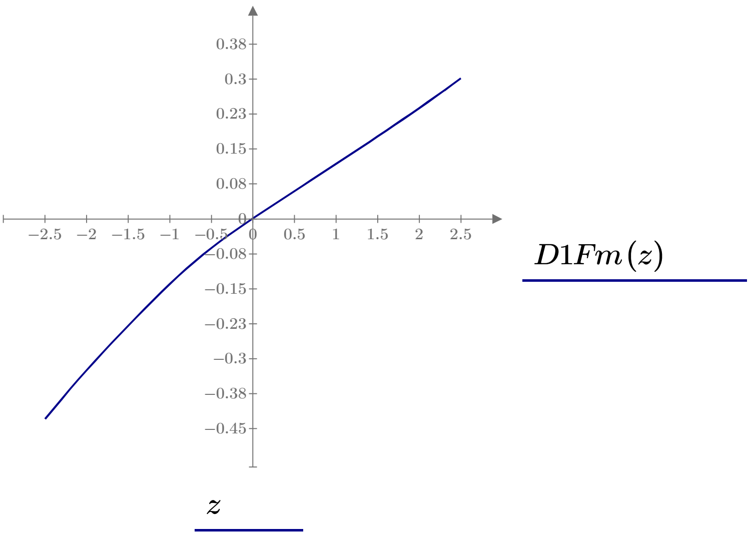


Fig 4

9.3.2 Geometry and equations of the progressive surface

We choose the example presented in table 3 in the columns 7 and 8 of the patent with the following definitions

.

$$rf := 8.75$$

$$a(z) := \left(\frac{1}{rf} + \left(\frac{\left(1 + e^{3 \cdot (z-1)}\right)^{-30}}{52.5} \right) \right)^{-1}$$

$$fs(z) := a(z) - Fm(z)$$

$$k(z) := 3 + \frac{7}{\left(1 + e^{(-3 \cdot (z + 1.8))}\right)}$$

$$b0(z) := fs(z) + \frac{1}{k(z)^2} \cdot \left(fs(z) - fs(z)^2 \cdot K(z) \cdot \sqrt{\left(1 + D1Fm(z)^2\right)} \right)$$

$$b1(z) := fs(z) - b0(z)$$

$$\rho s(\psi, z) := b0(z) + b1(z) \cdot \cos(k(z) \cdot \psi)$$

$$\rho(\psi,z) := \sqrt{\rho s(\psi,z)^2 + 2\ \rho s(\psi,z) \cdot (rf - a(z)) \cdot \cos(\psi) + (rf - a(z))^2}$$

$$\varphi(\psi,z) := \operatorname{atan}\left(\frac{\rho s(\psi,z) \cdot \sin(\psi)}{\rho s(\psi,z) \cdot \cos(\psi) + (rf - a(z))}\right)$$

$$xc(\psi,z) := \rho(\psi,z) \cdot \cos(\varphi(\psi,z))$$

$$yc(\psi,z) := \rho(\psi,z) \cdot \sin(\varphi(\psi,z))$$

$$x(\psi,z) := yc(\psi,z) \quad y(\psi,z) := rf - xc(\psi,z)$$

9.3.2.1 Fundamental Forms

Definition of the derivatives

$$r(\psi,z) := \begin{bmatrix} x(\psi,z) \\ y(\psi,z) \\ z \end{bmatrix}$$

$$x\psi(\psi,z) := \frac{d}{d\psi} x(\psi,z)$$

$$y\psi(\psi,z) := \frac{d}{d\psi} y(\psi,z)$$

$$r\psi(\psi,z) := \begin{bmatrix} x\psi(\psi,z) \\ y\psi(\psi,z) \\ 0 \end{bmatrix}$$

$$xz(\psi,z) := \frac{d}{dz} x(\psi,z)$$

$$yz(\psi,z) := \frac{d}{dz} y(\psi,z)$$

$$rz(\psi,z) := \begin{bmatrix} xz(\psi,z) \\ yz(\psi,z) \\ 1 \end{bmatrix}$$

$$x\psi\psi(\psi,z) := \frac{d^2}{d\psi^2} x(\psi,z)$$

$$y\psi\psi(\psi,z) := \frac{d^2}{d\psi^2} y(\psi,z)$$

$$r\psi\psi(\psi,z) := \begin{bmatrix} x\psi\psi(\psi,z) \\ y\psi\psi(\psi,z) \\ 0 \end{bmatrix}$$

$$xzz(\psi,z) := \frac{d^2}{dz^2} x(\psi,z)$$

$$yzz(\psi,z) := \frac{d^2}{dz^2} y(\psi,z)$$

$$rzz(\psi,z) := \begin{bmatrix} xzz(\psi,z) \\ yzz(\psi,z) \\ 0 \end{bmatrix}$$

$$x\psi z(\psi,z) := \frac{d}{dz} x\psi(\psi,z)$$

$$y\psi z(\psi,z) := \frac{d}{dz} y\psi(\psi,z)$$

$$r\psi z(\psi,z) := \begin{bmatrix} x\psi z(\psi,z) \\ y\psi z(\psi,z) \\ 0 \end{bmatrix}$$

1.Fundamental Form and unit normal vector

$$E(\psi,z) := r\psi(\psi,z) \cdot r\psi(\psi,z)$$

$$F(\psi,z) := r\psi(\psi,z) \cdot rz(\psi,z)$$

$$G(\psi,z) := rz(\psi,z) \cdot rz(\psi,z)$$

$$No(\psi,z) := \frac{(r\psi(\psi,z) \times rz(\psi,z))}{\|r\psi(\psi,z) \times rz(\psi,z)\|}$$

2. Fundamental Form

$$L(\psi,z) := r\psi\psi(\psi,z) \cdot No(\psi,z)$$

$$M(\psi,z) := r\psi z(\psi,z) \cdot No(\psi,z)$$

$$N(\psi,z) := rzz(\psi,z) \cdot No(\psi,z)$$

Principal Curvatures

$$H1(\psi,z) := E(\psi,z) \cdot N(\psi,z) + G(\psi,z) \cdot L(\psi,z) - 2\ F(\psi,z) \cdot M(\psi,z)$$

$$H2(\psi,z) := E(\psi,z) \cdot G(\psi,z) - F(\psi,z)^2$$

$$H3(\psi,z) := L(\psi,z) \cdot N(\psi,z) - M(\psi,z)^2$$

$$K1(\psi,z) := \frac{\left(H1(\psi,z) + \sqrt{H1(\psi,z)^2 - 4 \cdot H2(\psi,z) \cdot H3(\psi,z)}\right)}{2 \cdot H2(\psi,z)}$$

$$K2(\psi,z) := \frac{\left(H1(\psi,z) - \sqrt{H1(\psi,z)^2 - 4 \cdot H2(\psi,z) \cdot H3(\psi,z)}\right)}{2 \cdot H2(\psi,z)}$$

9.3.2.2 Mean optical power POW and Astigmatism AST

Due to the special choice of the coordinates (ρ, φ) the principal curvatures are negative. Thus, as the mean optical power in this example is positive, we insert in the formula for POW a minus sign. For the astigmatism this is not necessary, as K1>K2)

$$POW(\psi, z) := -52.5 \cdot \left(\frac{(K1(\psi, z) + K2(\psi, z))}{2} \right)$$

$$AST(\psi, z) := 52.5 \cdot (K1(\psi, z) - K2(\psi, z))$$

Check, that the principal meridian is an umbilical line

$$\psi := 0 \qquad z := -2.4, -2.2 .. 2.4$$

$$AST(\psi, z) = \begin{array}{r} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ \vdots \\ 24 \end{array} \begin{bmatrix} 4.17 \cdot 10^{-4} \\ 6.98 \cdot 10^{-4} \\ 1.58 \cdot 10^{-5} \\ 8.5 \cdot 10^{-5} \\ 6.41 \cdot 10^{-4} \\ 2.57 \cdot 10^{-3} \\ 1.12 \cdot 10^{-4} \\ 2.24 \cdot 10^{-3} \\ 9.4 \cdot 10^{-4} \\ 1.06 \cdot 10^{-3} \\ 1.88 \cdot 10^{-3} \\ \vdots \end{bmatrix}$$

Lines of constant mean optical power (Isopowerlines)

$$\psi min := 0 \qquad \psi max := 0.11 \cdot \pi \qquad n := 14 \qquad i := 0 .. n$$

$$\Delta\psi := \frac{\psi max - \psi min}{n} \qquad \psi_i := \psi min + i \cdot \Delta\psi$$

$zmin := -2.6 \qquad zmax := 2.6$

$$\Delta z := \frac{zmax - zmin}{n} \qquad z_i := zmin + i \cdot \Delta z$$

$$MPOW(\psi, z) := \left\| \begin{array}{l} \text{for } i \in 0..\text{last}(\psi) \\ \quad \left\| \begin{array}{l} \text{for } j \in 0..\text{last}(z) \\ \quad \left\| \begin{array}{l} MPow_{i,j} \leftarrow POW(\psi_i, z_j) + 0.03 \\ Mz_{i,j} \leftarrow z_j \\ Mx_{i,j} \leftarrow x(\psi_i, z_j) \end{array} \right. \end{array} \right. \\ \text{return} \begin{bmatrix} Mx \\ Mz \\ MPow \end{bmatrix} \end{array} \right.$$

a 0.03 D increment has been added to make the 9D line shine up in the isopower plot of *Mathcad*

$$ISOPOW := MPOW(\psi, z) = \begin{bmatrix} [15 \times 15] \\ [15 \times 15] \\ [15 \times 15] \end{bmatrix}$$

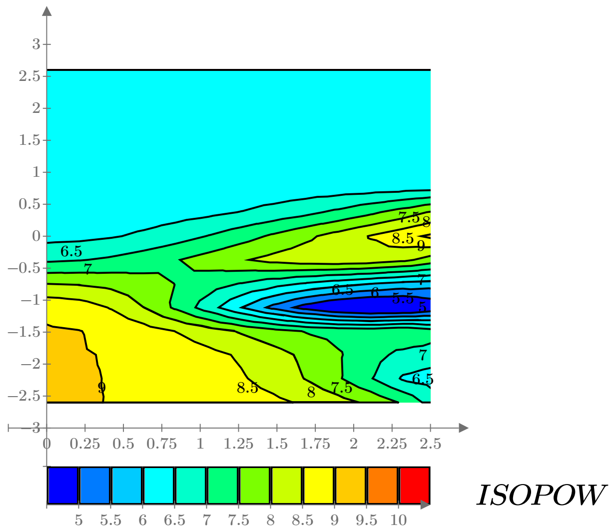


Fig 5

Lines of constant astigmatism (Isoastigmatism plot)

$\psi min := 0$ $\psi max := 0.11 \cdot \pi$ $n := 14$ $i := 0..n$

$$\Delta\psi := \frac{\psi max - \psi min}{n} \qquad \psi_i := \psi min + i \cdot \Delta\psi$$

$zmin := -2.6$ $zmax := 2.6$

$$\Delta z := \frac{zmax - zmin}{n} \qquad z_i := zmin + i \cdot \Delta z$$

$$MAST(\psi, z) := \begin{array}{l} \text{for } i \in 0..\text{last}(\psi) \\ \quad \text{for } j \in 0..\text{last}(z) \\ \quad\quad MAsti_{i,j} \leftarrow AST\left(\psi_i, z_j\right) \\ \quad\quad Mz_{i,j} \leftarrow z_j \\ \quad\quad Mx_{i,j} \leftarrow x\left(\psi_i, z_j\right) \\ \text{return } \begin{bmatrix} Mx \\ Mz \\ MAsti \end{bmatrix} \end{array}$$

$$ISOAST := MAST(\psi, z) = \begin{bmatrix} [15 \times 15] \\ [15 \times 15] \\ [15 \times 15] \end{bmatrix}$$

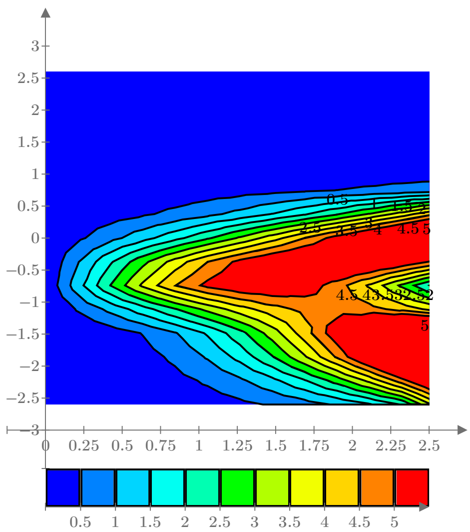


Fig 6

9.3.3 Analysis of design and calculation

Fig 6 shows a lens design with , as expected, large viewing zones for far and near vision . The far vision area is particularly free from aberrations. In the periphery the 0.5 D astigmatism line is bent downwards creating a strong gradient between far vision and progression. The maximum astigmatism value at x=2.5 cm is above 7 D.

If we now plot the figures of table 3 of the patent we get Fig 7 and 8

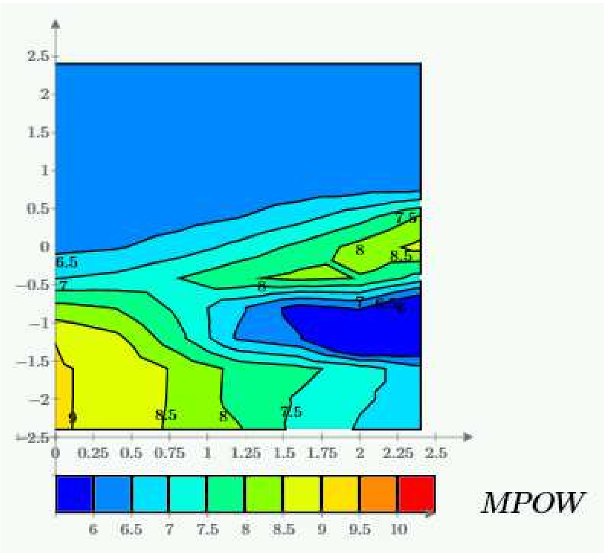


Fig 7

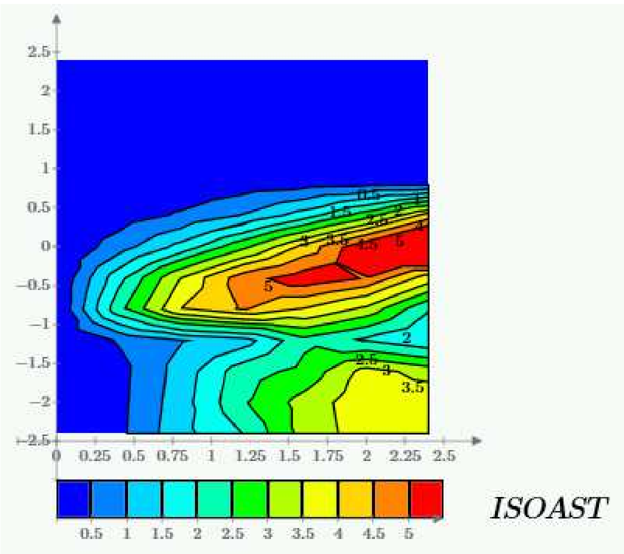


Fig 8

So the results of our calculations are clearly different from the figures of the patent. Particularly striking is the smaller astigmatism-free near vision portion in Fig 8. The comments under table 3 indicate that

$$k(z) = 3 + \frac{7}{\left(1 + e^{-3 \cdot (z+1.8)}\right)}$$

which we have used in our calculations . For strong negative z, this expression approaches k(z)≈3, which means a large NV-part in accordance with the results in Fig 6.

To calculate a design as in Fig 7 and 8 we have to start from a k-value in the upper part of the discussed k-range between 3 and 10. So for example the k(z)-function

$$k(z) = 9 + \frac{1}{\left(1 + e^{-3 \cdot (z+1.8)}\right)}$$

combined with a slightly softer power increase on the main meridian (following the add 2-trajectory) gives the isopower and isoastigmatism lines of Fig 9 and 10

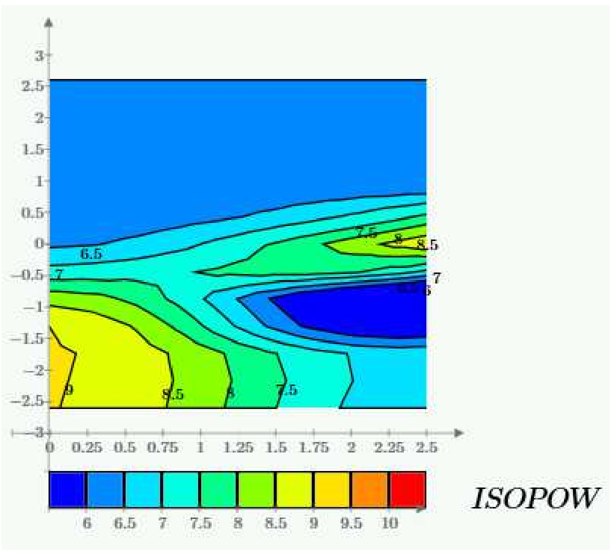


Fig 9

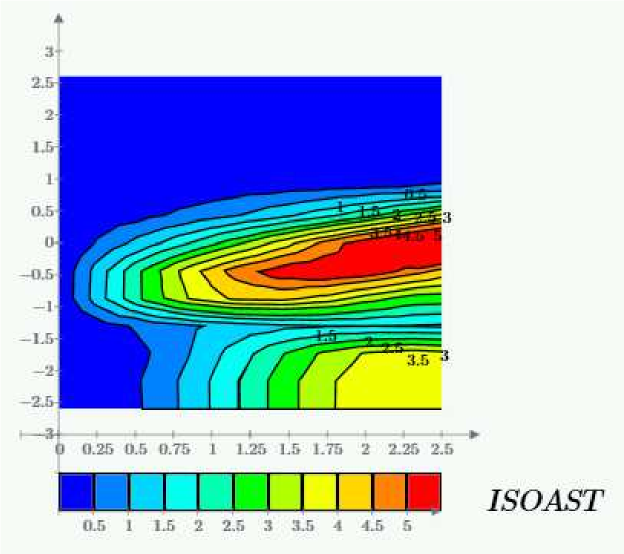


Fig 10

which are very close to the characteristics of Fig 7 and 8

9.4 Progressiv R, the commercial product

Now we consider the product which was sold with much success as the first progressive lens developed in Germany . The contour plots of this lens were published in the Deutsche Optikerzeitung [1] and in The Ophthalmic Optician [2]. They are reproduced here in Fig 11 a and b.

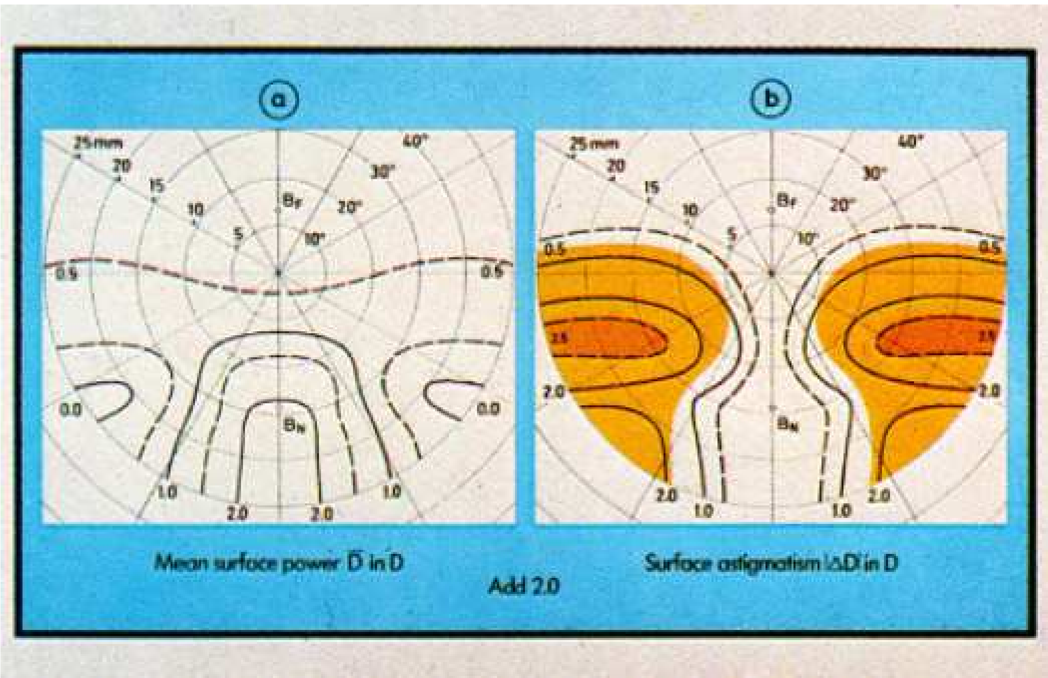


Fig 11 a and b

If we compare these plots with the design from the patent, represented in Fig 7 and 8 above, we see , even taking into account that table 3 (columns 7 and 8) of the patent describes a progressive with add 3, that the design of the market product is different from the example given in the patent.

So for example the ratio maximum astigmatism /add power for the market product is about 1.25, whereas for progressive surface of table 3 of the patent it is near to 1.7 , i.e. the isolines in Fig 11 a and b present a much softer product.

There is a big number of parameters in the formulas defining the progressive surface of Progressif R which influences the surface character and which will be discussed in chapter 9.5. We will learn, that rising the value of k(z) narrows the near vision zone and lowers the peripheral astigmatism. Other parameters influencing essentially the design are the progression length of the principal meridian and the exponential coefficient c0 in a(z).

In the following calculations we will vary some of these parameters and will obtain a design close to what has been published in Fig 11 a and b.

9.4.1 The principal meridian

We choose the main meridian of the patent claim 20, for the add power 1.0. The far vision radius r_f is 8 cm.

Integrating the differential equation for the curvature K(z) of the meridian

$$\frac{Fm''(z)}{\left(1+Fm'(z)^2\right)^{\frac{3}{2}}} = K(z) \qquad Fm(z) = x1(z)$$

$$x1' = x2$$

$$x2' = \left(1+x2^2\right)^{\frac{3}{2}} \cdot K(z)$$

$$K(z) := \frac{1}{8} + \frac{0.02}{0.525}\left(1-\left(1+e^{-2.29\cdot(z+2.19)}\right)^{-30}\right)$$

$$D(z,X) := \begin{bmatrix} X_1 \\ \left(1+X_1{}^2\right)^{\frac{3}{2}} \cdot K(z) \end{bmatrix}$$

$$X(z) = \begin{bmatrix} x1(z) \\ x2(z) \end{bmatrix}$$

$$u := 0.4057 \qquad v := 0.3316$$

$$init := \begin{bmatrix} u \\ v \end{bmatrix}$$

$$Zi := 2.5$$
$$Zf := -2.5$$
$$N := 100$$

$$sol := \mathrm{AdamsBDF}(init, Zi, Zf, N, D)$$

z x1=Fm(z) x1'=Fm'(z)

$$sol = \begin{bmatrix} 2.5 & 4.057 \cdot 10^{-1} & 3.316 \cdot 10^{-1} \\ 2.45 & 3.893 \cdot 10^{-1} & 3.243 \cdot 10^{-1} \\ 2.4 & 3.733 \cdot 10^{-1} & 3.171 \cdot 10^{-1} \\ 2.35 & 3.576 \cdot 10^{-1} & 3.099 \cdot 10^{-1} \\ 2.3 & 3.423 \cdot 10^{-1} & 3.027 \cdot 10^{-1} \\ 2.25 & 3.273 \cdot 10^{-1} & 2.956 \cdot 10^{-1} \\ 2.2 & 3.127 \cdot 10^{-1} & 2.885 \cdot 10^{-1} \\ 2.15 & 2.985 \cdot 10^{-1} & 2.815 \cdot 10^{-1} \\ & & \vdots \end{bmatrix}$$

Calculating the main meridian as a function of z : Fm(z)

Arranging the data in an ascending order of z

$$data := \mathrm{csort}(sol, 0) = \begin{bmatrix} -2.5 & 4.88 \cdot 10^{-1} & -4.21 \cdot 10^{-1} \\ -2.45 & 4.68 \cdot 10^{-1} & -4.11 \cdot 10^{-1} \\ -2.4 & 4.47 \cdot 10^{-1} & -4 \cdot 10^{-1} \\ -2.35 & 4.27 \cdot 10^{-1} & -3.9 \cdot 10^{-1} \\ -2.3 & 4.08 \cdot 10^{-1} & -3.8 \cdot 10^{-1} \\ -2.25 & 3.89 \cdot 10^{-1} & -3.7 \cdot 10^{-1} \\ -2.2 & 3.71 \cdot 10^{-1} & -3.6 \cdot 10^{-1} \\ -2.15 & 3.53 \cdot 10^{-1} & -3.51 \cdot 10^{-1} \\ & & \vdots \end{bmatrix}$$

$Z := data^{\langle 0 \rangle}$ $\quad X1 := data^{\langle 1 \rangle}$ $\quad X2 := data^{\langle 2 \rangle}$

$$S := \mathrm{cspline}(Z, X1)$$

$$Fm(z) := \mathrm{interp}(S, Z, X1, z)$$

$$S := \mathrm{cspline}(Z, X2)$$

$$D1Fm(z) := \mathrm{interp}(S, Z, X2, z)$$

9.4.2 Geometry and equations of the progressive surface

In the following calculations we will set k=13, c0=1.4 and the factor of the second term in a(z) equal to 2/2.5

$$rf := 8$$

$$a(z) := \left(\frac{1}{rf} + \left(2 \cdot \frac{\left(1 + e^{1.4 \cdot (z-1)}\right)^{-30}}{2.5 \cdot 52.5}\right)\right)^{-1}$$

$$fs(z) := a(z) - Fm(z)$$

$$k(z) := 13$$

$$b0(z) := fs(z) + \frac{1}{k(z)^2} \cdot \left(fs(z) - fs(z)^2 \cdot K(z) \cdot \sqrt{\left(1 + D1Fm(z)^2\right)}\right)$$

$$b1(z) := fs(z) - b0(z)$$

$$\rho s(\psi, z) := b0(z) + b1(z) \cdot \cos(k(z) \cdot \psi)$$

$$\rho(\psi, z) := \sqrt{\rho s(\psi, z)^2 + 2\ \rho s(\psi, z) \cdot (rf - a(z)) \cdot \cos(\psi) + (rf - a(z))^2}$$

$$\varphi(\psi,z) := \operatorname{atan}\left(\frac{\rho s(\psi,z)\cdot\sin(\psi)}{\rho s(\psi,z)\cdot\cos(\psi)+(rf-a(z))}\right)$$

$$xc(\psi,z) := \rho(\psi,z)\cdot\cos(\varphi(\psi,z))$$

$$yc(\psi,z) := \rho(\psi,z)\cdot\sin(\varphi(\psi,z))$$

$$x(\psi,z) := yc(\psi,z) \qquad\qquad y(\psi,z) := rf - xc(\psi,z)$$

9.4.2.1 Fundamental Forms

Definition of the derivatives

$$r(\psi,z) := \begin{bmatrix} x(\psi,z) \\ y(\psi,z) \\ z \end{bmatrix}$$

$$x\psi(\psi,z) := \frac{d}{d\psi}x(\psi,z)$$

$$y\psi(\psi,z) := \frac{d}{d\psi}y(\psi,z)$$

$$r\psi(\psi,z) := \begin{bmatrix} x\psi(\psi,z) \\ y\psi(\psi,z) \\ 0 \end{bmatrix}$$

$$xz(\psi,z) := \frac{d}{dz}x(\psi,z)$$

$$yz(\psi,z) := \frac{d}{dz}y(\psi,z)$$

$$rz(\psi,z) := \begin{bmatrix} xz(\psi,z) \\ yz(\psi,z) \\ 1 \end{bmatrix}$$

$$x\psi\psi(\psi,z) := \frac{d^2}{d\psi^2}x(\psi,z)$$

$$y\psi\psi(\psi,z) := \frac{d^2}{d\psi^2}y(\psi,z)$$

$$r\psi\psi(\psi,z) := \begin{bmatrix} x\psi\psi(\psi,z) \\ y\psi\psi(\psi,z) \\ 0 \end{bmatrix}$$

$$xzz(\psi,z) := \frac{d^2}{dz^2}x(\psi,z)$$

$$yzz(\psi,z) := \frac{d^2}{dz^2}y(\psi,z)$$

$$rzz(\psi,z) := \begin{bmatrix} xzz(\psi,z) \\ yzz(\psi,z) \\ 0 \end{bmatrix}$$

$$x\psi z(\psi,z) := \frac{d}{dz}x\psi(\psi,z)$$

$$y\psi z(\psi,z) := \frac{d}{dz}y\psi(\psi,z)$$

$$r\psi z(\psi,z) := \begin{bmatrix} x\psi z(\psi,z) \\ y\psi z(\psi,z) \\ 0 \end{bmatrix}$$

1. Fundamental Form and unit normal vector

$$E(\psi,z) := r\psi(\psi,z) \cdot r\psi(\psi,z)$$

$$F(\psi,z) := r\psi(\psi,z) \cdot rz(\psi,z)$$

$$G(\psi,z) := rz(\psi,z) \cdot rz(\psi,z)$$

$$No(\psi,z) := \frac{(r\psi(\psi,z) \times rz(\psi,z))}{\|r\psi(\psi,z) \times rz(\psi,z)\|}$$

2. Fundamental Form

$$L(\psi,z) := r\psi\psi(\psi,z) \cdot No(\psi,z)$$

$$M(\psi,z) := r\psi z(\psi,z) \cdot No(\psi,z)$$

$$N(\psi,z) := rzz(\psi,z) \cdot No(\psi,z)$$

Principal Curvatures

$$H1(\psi,z) := E(\psi,z) \cdot N(\psi,z) + G(\psi,z) \cdot L(\psi,z) - 2\ F(\psi,z) \cdot M(\psi,z)$$

$$H2(\psi,z) := E(\psi,z) \cdot G(\psi,z) - F(\psi,z)^2$$

$$H3(\psi,z) := L(\psi,z) \cdot N(\psi,z) - M(\psi,z)^2$$

$$K1(\psi,z) := \frac{\left(H1(\psi,z) + \sqrt{H1(\psi,z)^2 - 4 \cdot H2(\psi,z) \cdot H3(\psi,z)}\right)}{2 \cdot H2(\psi,z)}$$

$$K2(\psi,z) := \frac{\left(H1(\psi,z) - \sqrt{H1(\psi,z)^2 - 4 \cdot H2(\psi,z) \cdot H3(\psi,z)}\right)}{2 \cdot H2(\psi,z)}$$

9.4.2.2 Mean optical power POW and Astigmatism AST

$$POW(\psi, z) := -52.5 \cdot \left(\frac{(K1(\psi, z) + K2(\psi, z))}{2} \right)$$

$$AST(\psi, z) := 52.5 \cdot (K1(\psi, z) - K2(\psi, z))$$

Check that the principal meridian is an umbilical line

$$\psi := 0 \qquad z := -2.6, -2.4 .. 2.6$$

$$AST(\psi, z) = \begin{array}{r} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \vdots \\ 26 \end{array} \begin{bmatrix} 2.06 \cdot 10^{-2} \\ 8.66 \cdot 10^{-4} \\ 6.36 \cdot 10^{-4} \\ 5.71 \cdot 10^{-4} \\ 4.16 \cdot 10^{-4} \\ 2.34 \cdot 10^{-3} \\ 2.15 \cdot 10^{-3} \\ 1.35 \cdot 10^{-3} \\ \vdots \end{bmatrix}$$

Lines of constant mean optical power (Isopowerlines)

$$\psi min := -0.1 \cdot \pi \qquad \psi max := 0.1 \cdot \pi \qquad n := 14 \qquad i := 0 .. n$$

$$\Delta\psi := \frac{\psi max - \psi min}{n} \qquad \psi_i := \psi min + i \cdot \Delta\psi$$

$$zmin := -2.4 \qquad zmax := 2.4$$

$$\Delta z := \frac{zmax - zmin}{n} \qquad z_i := zmin + i \cdot \Delta z$$

$$MPOW(\psi,z):=\left\| \begin{array}{l} \text{for } i\in 0..\text{last}(\psi) \\ \quad \left\| \begin{array}{l} \text{for } j\in 0..\text{last}(z) \\ \quad \left\| \begin{array}{l} MPow_{i,j}\leftarrow POW\left(\psi_i,z_j\right) \\ Mz_{i,j}\leftarrow z_j \\ Mx_{i,j}\leftarrow x\left(\psi_i,z_j\right) \end{array}\right. \end{array}\right. \\ \text{return}\begin{bmatrix} Mx \\ Mz \\ MPow \end{bmatrix} \end{array}\right.$$

$$ISOPOW:=MPOW(\psi,z)=\begin{bmatrix}[15\times15]\\ [15\times15]\\ [15\times15]\end{bmatrix}$$

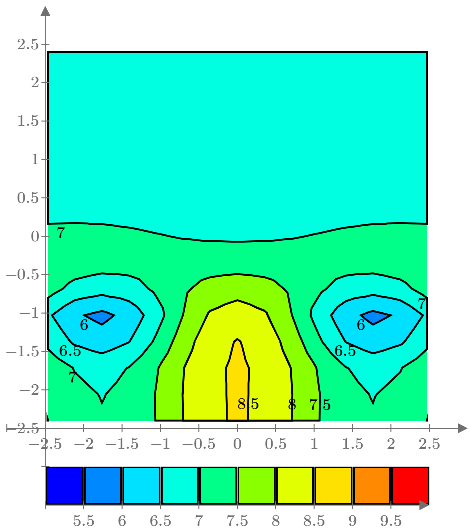


Fig 12

ISOPOW

Lines of constant astigmatism (Isoastigmatism plot)

$\psi min:=-0.1\cdot\pi$ $\quad\psi max:=0.1\cdot\pi$ $\quad n:=14$ $\quad i:=0..n$

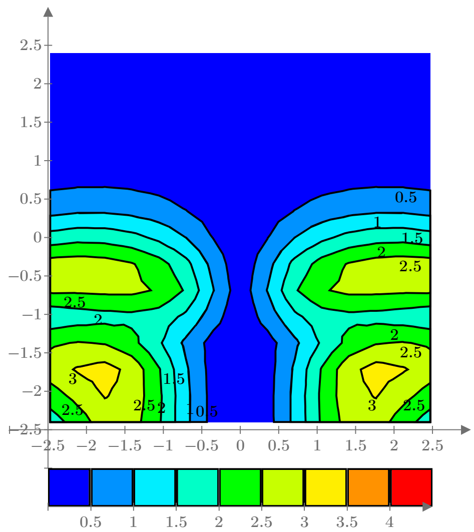
$$\Delta\psi:=\frac{\psi max-\psi min}{n}\qquad \psi_i:=\psi min+i\cdot\Delta\psi$$

$zmin:=-2.4$ $\quad zmax:=2.4$

$$\Delta z:=\frac{zmax-zmin}{n}\qquad z_i:=zmin+i\cdot\Delta z$$

$$MAST(\psi,z) := \left\| \begin{array}{l} \text{for } i \in 0..\text{last}(\psi) \\ \quad \left\| \begin{array}{l} \text{for } j \in 0..\text{last}(z) \\ \quad \left\| \begin{array}{l} MAsti_{i,j} \leftarrow AST\left(\psi_i, z_j\right) \\ Mz_{i,j} \leftarrow z_j \\ Mx_{i,j} \leftarrow x\left(\psi_i, z_j\right) \end{array} \right. \end{array} \right. \\ \text{return} \begin{bmatrix} Mx \\ Mz \\ MAsti \end{bmatrix} \end{array} \right.$$

$$ISOAST := MAST(\psi,z) = \begin{bmatrix} [15 \times 15] \\ [15 \times 15] \\ [15 \times 15] \end{bmatrix}$$



$ISOAST$ Fig 13

9.4.3 Analysis of design and calculations

The design in Fig 12 and 13 is almost identical to the isolines plots in Fig 11 a and b, regarding the characteristical structure as well as the dimensions, like widths of the different viewing zones. Thus the specific parameter choice in the calculations above is one option to get a design close to the lens, Rodenstock marketed as Progressiv R. But taking into account the big number of parameters in the formulas of the patent, there are certainly other possible combinations to obtain about the same result.

Comparing now the properties of Progressiv R, Varilux 1 and Varilux 2 , we have to take into account that Varilux 1 is a lens of the first generation and that the goal was to construct a surface whith bifocal-like features. Accordingly Varilux 1 is distinguished by large aberrationfree NV and FV regions, accepting however a high amount of astigmatism and strong power/astigmatism gradients in the intermediate periphery.
Varilux 2 and Progressiv R benefit from the large scale market feedback from the Varilux 1 wearers. The Varilux 2 surface, with an astigmatism maximum value of about the add power and its considerable reduction of distortion, improved significantly the visual comfort, which was the breakthrough on the market. As the Varilux 2 geometry is totally aspherized, the surface astigmatism extends into the FV periphery. The Progressiv R design reduces this lateral FV astigmatism, has a width of the near vision zone in-between the Varilux 1 and the Varilux 2 design and shows a reduced peripheral astigmatism in the progression of a little above the add power.

The challenge for the Progressiv R development was to design a progressive surface with a visual comfort competitive with Varilux 2 and supporting the (mainly commercial) argument, that the lens power could be ordered exactly to prescripton. Extended double blind wearer tests confirmed, that Progressiv R was appreciated by its excellent overall performance and its remarkable far vision quality [2].

9.5 The structure of the progressive surface with variable periodicity

9.5.1 Reasoning for the functions k(z) and a(z)

To introduce the discussion let us have a look how the plots evolve, when we vary the parameters of k(z) and a(z). For the add power 3 we choose the cornerpoints of the two-dimensional (a,k)-range analyzed in the tables 1 to 3 in the columns 7 and 8 of the patent . In order to demonstrate the characteristics and differences, in the Fig 14-17 below we use the isoastigmatism-plots, which illustrate the differences more clearly. (For improved clarity in these diagrams the spacing between the isolines is 1 D starting with the 0.5 D curve.)

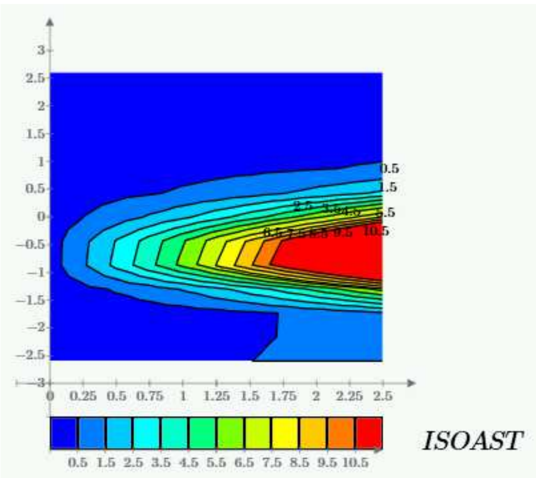


Fig 14: $a(z)=r_f$, $k(z)=3$

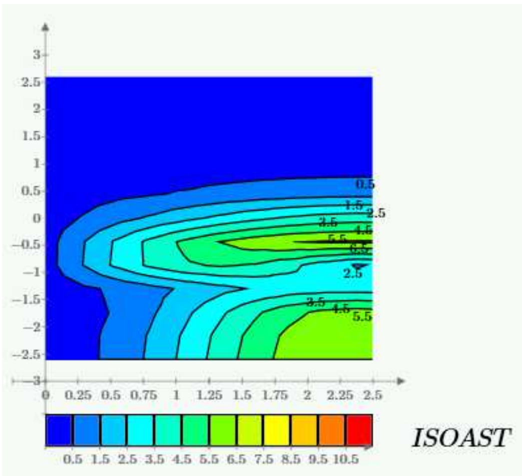


Fig 15: $a(z)=r_f$, $k(z)=10$

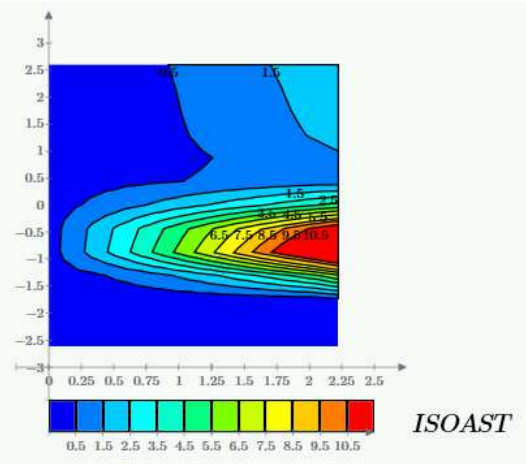


Fig 16: a(z)=rn, k(z)=3

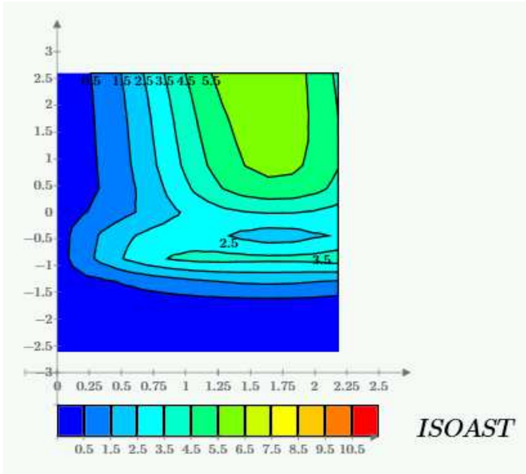


Fig 17: a(z)=rn, k(z)=10

where r_f is the radius of curvature in the far vision reference point and r_n the radius of curvature in the near vison reference point. In this example are r_f=8 cm and r_n=5.49 cm.

The combination in Fig 15 represents a reasonable design with good quality of the far vision zone and the lateral astigmatism, but the near vision part is rather narrow. Concerning the performance of the near vision zone the isoplot in Fig 16 is certainly the best design, but here the far vision part is rather poor and the peripheral astigmatism is marked by a strong gradient.

So a reasonable solution could be to calculate the lower part of the lens according to a combination close to (a=r_n, k=3), the the upper part using (a=r_f, k=10) and optimizing the sections of the intermediate region according to a continuous transition between these two (a,k)-couples.

So G. Guilino and R. Barth proposed first an auxiliary coordinate system with a curvilinear cylinder axis defined by its distance a(z) to the z-axis of the (x,y,z)-coordinate system:

$$a(z) = \left(\frac{1}{rf} + \frac{\left(1 + e^{c0\,(z - d0)}\right)^{-m0}}{\frac{3}{A} \cdot 52.5} \right)^{-1}$$

where particularly c0=3, d0=1, m0=30

and second a periodicity function k(z)

$$k(z) = 3 + \frac{7}{\left(1 + e^{-3 \cdot (z + 1.8)}\right)}$$

9.5.2 Why periodic horizontal sections?

Guilino and Barth explain their choice of the design of variable periodicity [1] with the fact, that periodic functions will remain limited below certain thresholds and so also characteristic surface properties, as astigmatism in the periphery, will not increase beyond certain thresholds. In order to check this hypothesis let us analyze in the following section 3 examples.

First Example : a(z) and k(z) are constant

We choose the example above $a(z)=r_f$ and $k(z)= 10$ and analyse how the isolines behave, if we consider the more distant design periphery. Fig. 18 shows a clear periodicty of the design pattern . The first segment is symmetric to the vertical axis at x about 2.3 cm and a second segment starts at x about 4.6 cm . The vertical at this place is a second ombilic line.

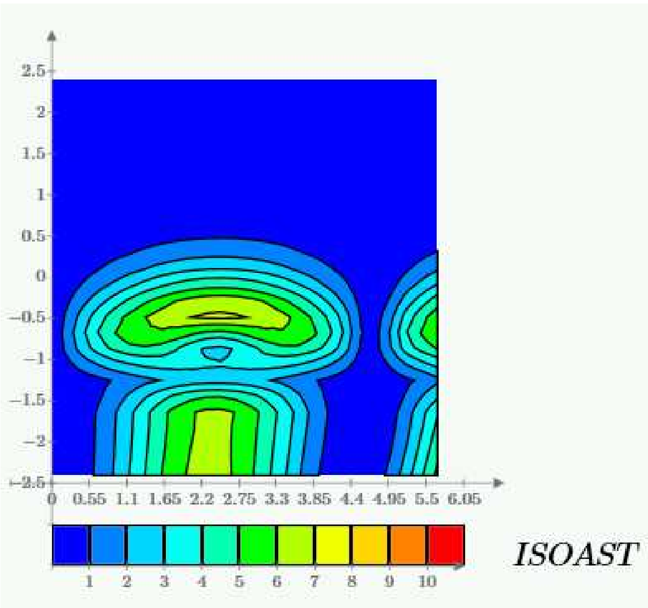


Fig 18 : $a(z)=r_f$, $k(z)=10$

The reason for this periodicity with a wavelength of about 4.6 cm is the curvature modulation of the horizontal sections. Fig 19a to 19c show the optical power p_r in D for the horizontal sections for z=-1.6 (near vision), z=-0.8 (intermediate vision) and z=1.0 (far vision)

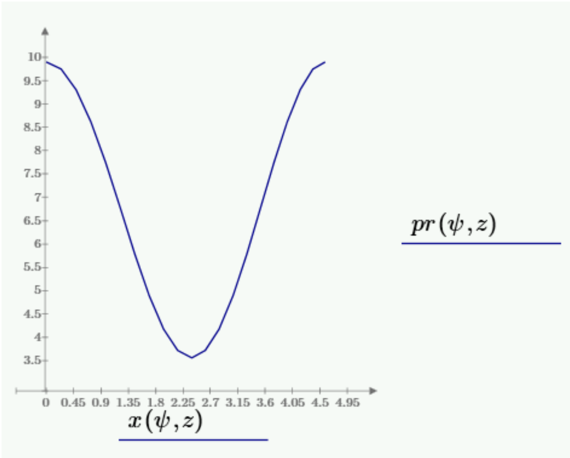


Fig. 19 a: add 3, a=rf, k=10, z=-1.6

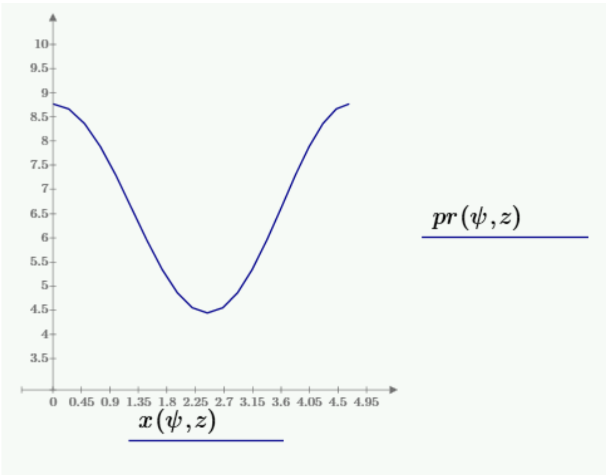


Fig 19 b: add 3, a=rf, k=10, z=-0.8

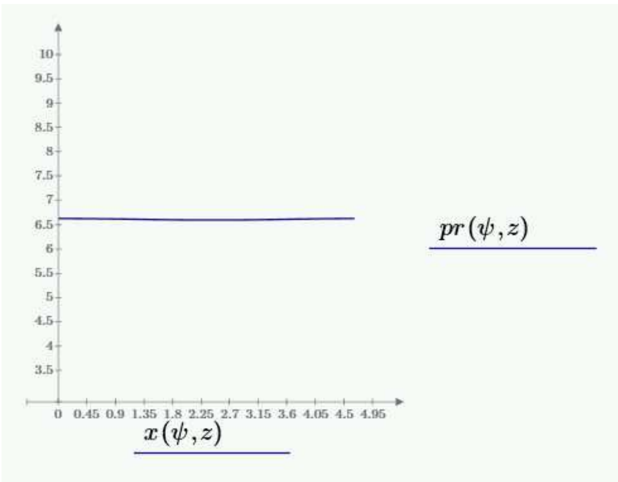


Fig 19 c: add 3,a=rf, k=10, z=1.0

As expected, considering the mathematical approach in chapter 9.2, the curvature of the horizontal sections follows a a sine-curve. From the maximum at x=0 the curvature decreases until a minimum at x=2.3/2.45 cm (slightly rising from NV to FV), from where it grows to a maximum at x=4.6/4.9 cm, corresponding to the periodicity of the isolines design pattern.

This lateral curvature decrease is strong in the near vision portion, becoming smaller in the progression and approaching zero in the far vision.

Above z= 1cm the sections are pure circles.

Second example: the patent

We take the formulas and figures of table 3 of the columns 7 and 8 of the patent

$$rf = 8.75 \quad \text{A=3} \quad c_0 = 3 \quad d_0 = 1 \quad m_0 = 30$$

$$a(z) = \left(\frac{1}{rf} + \frac{\left(1 + e^{c0\,(z - d0)}\right)^{-m0}}{\frac{3}{A} \cdot 52.5} \right)^{-1}$$

$$k(z) = 3 + \frac{7}{\left(1 + e^{-3 \cdot (z + 1.8)}\right)}$$

Fig 20 a, 20 b and 20 c show the optical power p_r calculated for the horizontal sections at z=-1.6,-0.8 and 1.0 cm (k(z) =3+7 is used as abbreviation for the extended formula above)

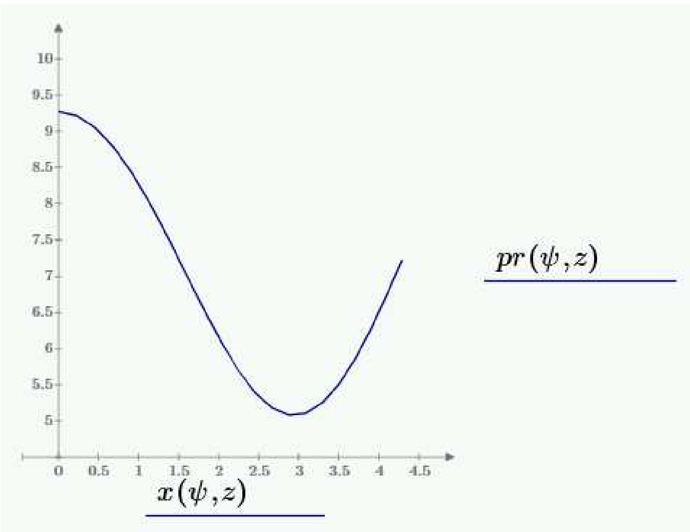


Fig 20 a: add 3, k(z)=3+7, z=-1.6

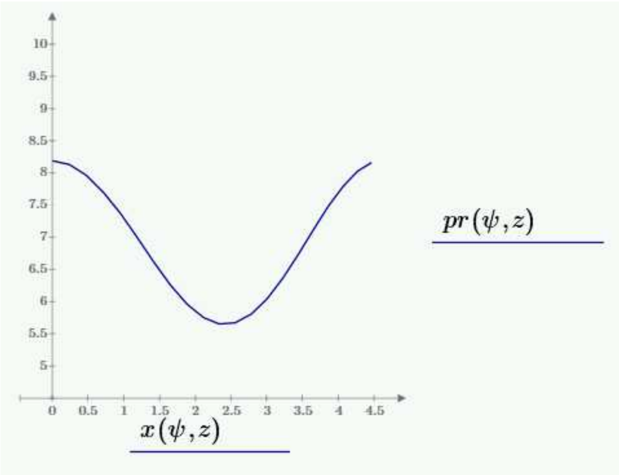


Fig 20 b: add 3, k(z)=3+7, z=-0.8

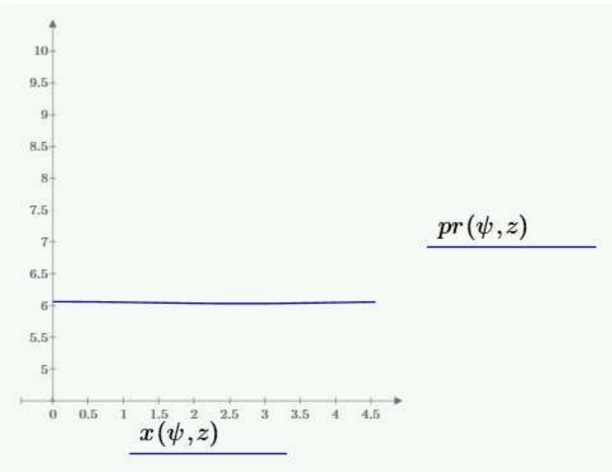


Fig 20 c: add 3, k(z)=3+7, z=1.0

The periodicity k(z) is growing from the near vision point z=-1.6 to the far vision point z=1.0 from 7.5 to 10. At the bottom of the lens (z<-2.5 cm) the curvature minimum is situated at x> 4 cm , i.e. outside the lens surface, and in the far vision region the position of the (very flat) minimum is at about x=2.5 cm i.e. at an viewing angle of about 40°. For k(z)=3+7, within the large field of view of 40°, the curvature of the horizontal sections is only decreasing. This lateral curvature decrease is strong in the near vision portion, becoming smaller in the progression and approaching zero in the far vision field.

Above z= 1 cm the sections are pure circles. Nowhere in the far vision part the horizontal section starts with a curvature increase at the main meridian as it as the case for Varilux 2.

The periodicity of the horizontal sections does not show up in the pattern of the isolines. If we extend the calculations until x-values of 4 to 5 cm we encounter astigmatism values of 20 and more diopters and no periodicity of the design can be detected.

Third example: the market product

The last example, which we will analyze a little closer, is the design which approximates the product which had been sold on the market. The horizontal sections for z=-1.6, z=-0.8 and z=1.0 are presented in Fig 21 a, 21 b and 21 c.

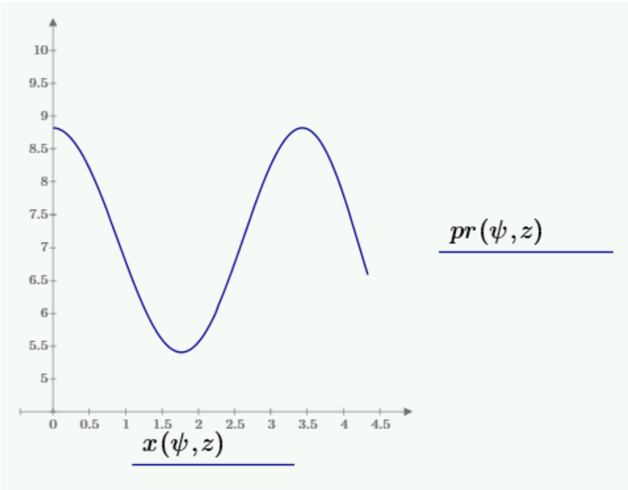


Fig 21 a: add 2, k(z)=13, z=-1.6

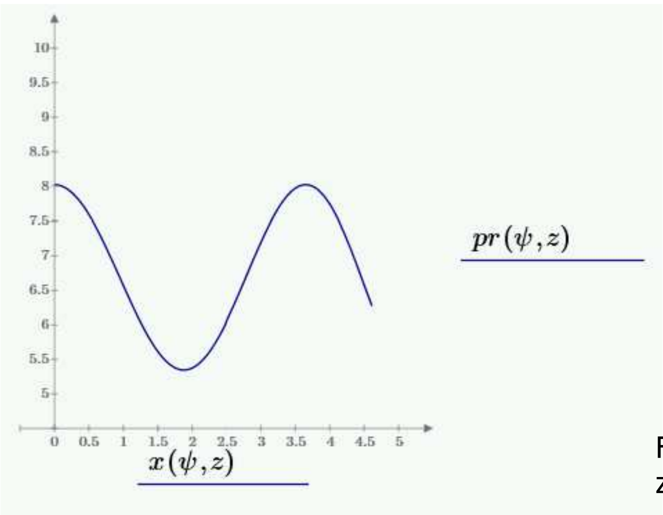


Fig 21 b: add 2, k(z)=13, z=-0.8

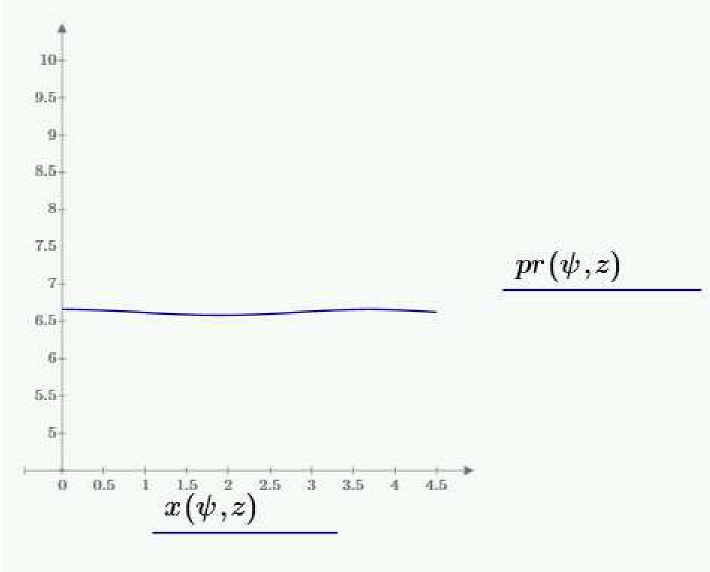


Fig 21 c : add 2, k(z)=13, z=1.0

For k=13 the modulation wavelength for the three horizontal sections is nearly constant , i.e. 3.5 cm for k=-1.6 and 3.8 cm for k=-0.8 and 1.0 . The amplitude of the sine-curve is somewhat smaller than for k(z)=3+7 . For $z > 1.5$ the horizontal sections are pure circles .
Fig 22 shows the isoastigmatism pattern, if the calculations are extended until x=5.5 cm. The astigmatism increases until a vertical "symmetry-axis" at x=1.8, then it decreases and starts growing again at x about 3.3 cm.

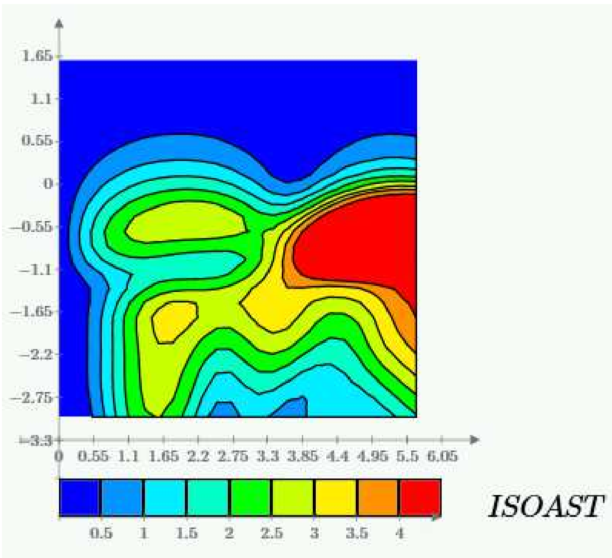


Fig 22

9.5.3 Conclusion

When a and k are both constant the design shows a periodic structure of the isolines , the periodicity being identical with that of the curvature modulation of the horizontal sections.

If we choose k variable with z, a small initial value of 3 to 5 for negative z means a broad NV-part combined with a rather high amount of lateral astigmatism in the lower part of the lens . Increasing this initial value means that the near vison part will become smaller and the aberrations in the periphery of the near vision and lower intermediate portion will diminish. The quality of the upper intermediate portion will be improved by increasing the total k-value (10 to 15) for high positive z. A combination of a low initial k-value with a high total k-value reduces neatly the extension of the island of negative power in the periphery of the powerplot.

A strong influence on the design characteristics has the a(z) parameter c0. A c0 value of 3 represents a very sudden transition from the intermediate vision to the far vision zone . If the c0 value is reduced, the curvature of the a(z) axis of the auxiliary coordinate system becomes flatter, i.e. the transition of the function a(z) to its asymptotic values becomes softer. This means a smaller gradient of astigmatism and power between FV and IV zone and a symmetry in the astigmatism pattern with a decrease of the astigmatism in the lens periphery, which allows the lens designer to limit the lateral aberrations.

Thus the combination of a high total k-value (periodicity-wavelength in the order of the lens radius, see Fig 21) with a low c0 of about 1.5 is one way to reach low peripheral astigmatism as in the market design of Progressif R. Higher k- values cause however smaller NV parts, but as visual acuity requirements for near vision work are lower, the usable near vision portion is determined by the 1 D-and 2 D-isoastigmatism lines and remains relatively large (see also the NV part outside the yellowish- shadowed region in Fig 11b).

References

1. Günther Guilino and Rudolf Barth: Neue progressive Flächen. Deutsche Optikerzeitung, no. 11, 1980

2. Günther Guilino and Werner Köppen: Progressiv R: The new progressive lens from Rodenstock, The Ophthalmic Optician, March 13, 1982